

DISCRETE-TIME COHEN-GROSSBERG NEURAL NETWORKS WITH TRANSMISSION DELAYS AND IMPULSES

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ABSTRACT. A discrete-time analogue is formulated for an impulsive Cohen-Grossberg neural network with transmission delay in a manner in which the global exponential stability characteristics of a unique equilibrium point of the network are preserved. The formulation is based on extending the existing semi-discretization method that has been implemented for computer simulations of neural networks with linear stabilizing feedback terms. The exponential convergence in the p -norm of the analogue towards the unique equilibrium point is analysed by exploiting an appropriate Lyapunov sequence and properties of an M -matrix. The main result yields a Lyapunov exponent that involves the magnitude and frequency of the impulses. One can use the result for deriving the exponential stability of non-impulsive discrete-time neural networks, and also for simulating the exponential stability of impulsive and non-impulsive continuous-time networks.

1. Introduction

Numerical (or computer) simulations of continuous-time neural networks governed by differential equations involving transmission delays (*i.e.*, discrete or fixed delays, time-varying delays and distributed delays) have been developed steadily over the years [14]–[21], [23]–[34]. These simulations are produced by discrete-time analogues formulated from semi-discretization method, in a manner in which the dynamical characteristics—such as the sets of equilibria and their stability characteristics—of the networks are preserved. The successful implementation of the analogues is largely due to the presence of feedback terms and denominator functions that correspond to the linear stabilizing feedback terms of the continuous-time networks. With a similar architectural design, the method has been applied further by Akça *et al.* [1], Covachev *et al.* [7],

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Gopalsamy [11] and Mohamad [22] for simulating the dynamical characteristics of continuous-time neural networks that are subject to impulsive state displacements at fixed moments of time.

Cohen-Grossberg neural network [6] and its various generalizations with or without transmission delays and impulsive state displacements have been the subject of intense investigation recently [2, 4, 5, 27, 31, 32]. In a Cohen-Grossberg neural network model, the feedback terms consist of amplification and stabilizing functions which are generally nonlinear. These terms provide the model with a special kind of generalization wherein many neural network models that are capable for content addressable memory such as additive neural networks, cellular neural networks and bidirectional associative memory networks and also biological models such as Lotka-Volterra models of population dynamics are included as special cases.

Despite the generality, the nonlinear feedback terms of the networks are not cohesive with a direct extension of the existing semi-discretization method for obtaining the corresponding discrete-time analogues. In pursuance of the extension, a systematic approach of the semi-discretization method is given in this paper for obtaining the corresponding analogue of a Cohen-Grossberg neural network with distributed transmission delays and impulsive state displacements at fixed moments of time. The dynamical characteristics of the analogue in the general norm $\|\cdot\|_p$ ($p \geq 1$ is a real number) are subsequently investigated by exploiting an appropriate Lyapunov sequence and properties of a monotone matrix (i.e., M -matrix), and this leads to obtaining a Lyapunov exponent expressed in the form of a relation involving the magnitude and frequency of the impulses. A corollary is given to demonstrate the effectiveness of the main result.

2. Impulsive continuous-time Cohen-Grossberg neural network

The impulsive continuous time neural network consists of m elementary processing units (or neurons) whose state variables x_i ($i = \overline{1, m}$) are governed by

$$\frac{dx_i(t)}{dt} = a_i(x_i(t)) \left[-b_i(x_i(t)) + \sum_{j=1}^m c_{ij} f_j(x_j(t - \tau_{ij})) + \sum_{j=1}^m d_{ij} \int_0^{\infty} K_{ij}(s) g_j(x_j(t - s)) ds + I_i \right], \quad t > t_0, \quad t \neq t_k, \quad (1)$$

$$\Delta x_i(t_k) = r_{ik}(x_i(t_k^-)), \quad i = \overline{1, m}, \quad k \in \mathbb{N} = \{1, 2, 3, \dots\},$$

with initial values prescribed by piecewise-continuous functions $x_i(s) = \phi_i(s)$ which are bounded for $s \in (-\infty, t_0]$. In (1) $a_i(x_i)$ denotes an amplification function; $b_i(x_i)$ denotes an appropriate function which supports the stabilizing (or negative) feedback term $-a_i(x_i)b_i(x_i)$ of the unit i ; $f_j(x_j)$, $g_j(x_j)$ denote activation functions; the parameters c_{ij} , d_{ij} are real numbers that represent the weights (or strengths) of the synaptic connections between the j th unit and the i th unit; the real constant I_i represents an input signal introduced from outside the network to the i th unit; τ_{ij} are nonnegative real numbers whose presence indicates the delayed transmission of signals at time $t - \tau_{ij}$ from the j th unit to the unit i ; and the delay kernels K_{ij} incorporate the fading past effects (or fading memories) of the j th unit on the i th unit; $\Delta x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$ denote impulsive state displacements characterized by the nonlinear functions $r_{ik}(x_i(t_k^-))$ at fixed moments of time t_k , $k \in \mathbb{N}$. Here it is assumed that $x_i(t_k^+) = \lim_{t \rightarrow t_k^+} x_i(t)$ and $x_i(t_k^-) = \lim_{t \rightarrow t_k^-} x_i(t)$, and the sequence of times $\{t_k\}_{k=1}^\infty$ satisfies $t_0 < t_1 < t_2 < \dots < t_k \rightarrow \infty$ as $k \rightarrow \infty$ and $\Delta t_k = t_k - t_{k-1} \geq \theta$, where $\theta > 0$ denotes the minimum time interval between successive impulses. In other words, the value $\theta > 0$ means that the impulses do not occur too often, but $\theta = \infty$ means that the network (1) is free of impulses.

The assumptions that accompany the impulsive network (1) are given as follows:

A₁: The amplification functions $a_i: \mathbb{R} \rightarrow \mathbb{R}^+$ are continuous and bounded in the sense that

$$0 < \underline{a}_i \leq a_i(u) \leq \bar{a}_i \quad \text{for } u \in \mathbb{R}, \quad i = \overline{1, m}.$$

A₂: The stabilizing functions $b_i: \mathbb{R} \rightarrow \mathbb{R}$ are continuous and monotone increasing, namely,

$$0 < \underline{b}_i \leq \frac{b_i(u) - b_i(v)}{u - v} \quad \text{for } u \neq v, \quad u, v \in \mathbb{R}, \quad i = \overline{1, m}.$$

A₃: The activation functions $f_j, g_j: \mathbb{R} \rightarrow \mathbb{R}$ with $f_j(0) = g_j(0) = 0$ are Lipschitz continuous in the sense of

$$L_j^f = \sup_{u \neq v} \left| \frac{f_j(u) - f_j(v)}{u - v} \right|, \quad L_j^g = \sup_{u \neq v} \left| \frac{g_j(u) - g_j(v)}{u - v} \right| \quad \text{for } u, v \in \mathbb{R}, \quad i = \overline{1, m},$$

where L_j^f, L_j^g denote positive constants.

A₄: For the impulse functions $r_{ik}: \mathbb{R} \rightarrow \mathbb{R}$ there exist positive numbers γ_{ik} such that

$$\gamma_{ik} = \sup_{u \neq v} \left| \frac{r_{ik}(u) - r_{ik}(v)}{u - v} \right| \quad \text{for } u, v \in \mathbb{R}, \quad k \in \mathbb{N}, \quad i = \overline{1, m}.$$

A₅: The delay kernels $K_{ij} : [0, \infty) \rightarrow [0, \infty)$ are piecewise continuous functions that satisfy

$$\int_0^{\infty} K_{ij}(s) ds = \kappa_{ij} \quad \text{and} \quad \int_0^{\infty} K_{ij}(s)e^{\nu_0 s} ds < \infty,$$

where κ_{ij} denote nonnegative constants and ν_0 is some positive number.

Under these assumptions and the given initial conditions, there is a unique solution of the impulsive network (1). The solution is a vector

$$x(t) = (x_1(t), x_2(t), \dots, x_m(t))^T$$

in which $x_i(t)$ are piecewise continuous for $t \in (t_0, \beta)$, where $\beta > t_0$ is some positive number, possibly ∞ , such that the limits $x_i(t_k^+)$ and $x_i(t_k^-)$ exist and $x_i(t)$ are differentiable for $t \in (t_{k-1}, t_k) \subset (t_0, \beta)$. An equilibrium point of the impulsive network (1) is denoted by $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ whereby the components x_i^* are governed by the algebraic system

$$b_i(x_i^*) = \sum_{j=1}^m c_{ij} f_j(x_j^*) + \sum_{j=1}^m d_{ij} \kappa_{ij} g_j(x_j^*) + I_i, \quad i = \overline{1, m}, \quad (2)$$

and satisfy the equalities

$$r_{ik}(x_i^*) = 0, \quad k \in \mathbb{N}, \quad i = \overline{1, m}. \quad (3)$$

3. Formulation of an impulsive discrete-time analogue

For a non-impulsive neural network model with linear stabilizing feedback terms given by $-\hat{b}_i x_i(t)$, that is, the terms are obtained from (1) by letting $a_i(x_i(t)) = 1$ and $b_i(x_i(t)) = \hat{b}_i x_i(t)$, where \hat{b}_i are positive constants, the application of the existing semi-discretization method [25] yields a discrete-time analogue with corresponding feedback terms given by $e^{-\hat{b}_i h} x_i(n)$, where h denotes a fixed time-step of the discretization. These terms together with the associated denominator functions given by $\hat{\psi}_i(h) = \frac{1 - e^{-\hat{b}_i h}}{\hat{b}_i}$ provide the analogue with features that are conducive for preserving the dynamical characteristics of the continuous-time model, without requiring to restrict the magnitude of the time-step and introduce additional restrictions not envisaged in the continuous-time models.

Till now, the semi-discretization model has not been exploited for obtaining a discrete-time analogue of Cohen-Grossberg neural network mainly due to the nonlinearity of the feedback terms $-a_i(x_i)b_i(x_i)$. An appropriate extension of the

method is presented here. We begin by rewriting the differential system in (1) as

$$\begin{aligned} \frac{dx_i(t)}{dt} = & -\beta_i x_i(t) + \beta_i x_i(t) + a_i(x_i(t)) \left[-b_i(x_i(t)) + \sum_{j=1}^m c_{ij} f_j(x_j(t - \tau_{ij})) \right. \\ & \left. + \sum_{j=1}^m d_{ij} \int_0^\infty K_{ij}(s) g_j(x_j(t-s)) ds + I_i \right], \quad i = \overline{1, m}, \quad t > t_0, \quad t \neq t_k, \end{aligned} \quad (4)$$

where $\beta_i = \underline{a}_i \underline{b}_i > 0$. Let the value $h \in (0, \theta)$ be fixed, and $n_0 = \lfloor \frac{t_0}{h} \rfloor$, $n = \lfloor \frac{t}{h} \rfloor$, $\sigma_{ij} = \lfloor \frac{\tau_{ij}}{h} \rfloor$ and $\ell = \lfloor \frac{s}{h} \rfloor$, where $\lfloor r \rfloor$ denotes the greatest integer contained in the real number r . On any interval $[nh, (n+1)h)$ not containing a moment of impulse effect t_k the equation (4) can be approximated by equations with constant arguments of the form

$$\begin{aligned} \frac{dx_i(t)}{dt} = & -\beta_i x_i(t) + \beta_i x_i\left(\left[\frac{t}{h}\right]h\right) \\ & + a_i\left(x_i\left(\left[\frac{t}{h}\right]h\right)\right) \left[-b_i\left(x_i\left(\left[\frac{t}{h}\right]h\right)\right) + \sum_{j=1}^m c_{ij} f_j\left(x_j\left(\left[\frac{t}{h}\right]h - \left[\frac{\tau_{ij}}{h}\right]h\right)\right) \right. \\ & \left. + \sum_{j=1}^m d_{ij} \sum_{\lfloor \frac{s}{h} \rfloor=1}^\infty \mathcal{K}_{ij}\left(\left[\frac{s}{h}\right]h\right) g_j\left(x_j\left(\left[\frac{t}{h}\right]h - \left[\frac{s}{h}\right]h\right)\right) + I_i \right], \quad i = \overline{1, m}, \end{aligned} \quad (5)$$

with

$$\begin{aligned} \left[\frac{t}{h}\right]h = nh \rightarrow t, \quad \left[\frac{\tau_{ij}}{h}\right]h = \sigma_{ij}h \rightarrow \tau_{ij}, \quad \left[\frac{s}{h}\right]h = \ell h \rightarrow s, \\ \sum_{\lfloor \frac{s}{h} \rfloor=1}^\infty \mathcal{K}_{ij}\left(\left[\frac{s}{h}\right]h\right) g_j\left(x_j\left(\left[\frac{t}{h}\right]h - \left[\frac{s}{h}\right]h\right)\right) \rightarrow \int_0^\infty K_{ij}(s) g_j(x_j(t-s)) ds \end{aligned}$$

for a fixed time t as $h \rightarrow 0$. Moreover,

A₅': The delay kernels $\mathcal{K}_{ij}: \mathbb{N} \rightarrow [0, \infty)$ ($i, j = \overline{1, m}$) satisfy

$$\sum_{\ell=1}^\infty \mathcal{K}_{ij}(\ell) = \kappa_{ij} \quad \text{and} \quad \sum_{\ell=1}^\infty \mathcal{K}_{ij}(\ell) e^{\nu \ell h} < \infty,$$

where the positive real number $\nu = \nu(h)$ for a given $h \in (0, \theta)$ is related to the positive number ν_0 (cf. assumption **A₅**) by $\nu \rightarrow \nu_0$ (from below) as $h \rightarrow 0$.

For simplicity, we write system (5) as

$$\begin{aligned} \frac{dx_i(t)}{dt} = & -\beta_i x_i(t) + \beta_i x_i(n) \\ & + a_i(x_i(n)) \left[-b_i(x_i(n)) \sum_{j=1}^m + \sum_{j=1}^m c_{ij} f_j(x_j(n - \sigma_{ij})) \right. \\ & \left. + \sum_{j=1}^m d_{ij} \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) g_j(x_j(n - \ell)) + I_i \right] \end{aligned} \quad (6)$$

for $i = \overline{1, m}$, $t \in [nh, (n+1)h)$, $t \neq t_k$, $n = n_0, n_0 + 1, \dots$, wherein the notation $w(n) \equiv w(nh)$ has been adopted. Upon integrating (6) over the interval $[nh, (n+1)h)$, one obtains a discrete analogue of the differential system in (1) given by

$$\begin{aligned} x_i(n+1) = & e^{-\beta_i h} x_i(n) + \psi_i(h) \left\{ \beta_i x_i(n) + a_i(x_i(n)) \left[-b_i(x_i(n)) \right. \right. \\ & \left. \left. + \sum_{j=1}^m c_{ij} f_j(x_j(n - \sigma_{ij})) + \sum_{j=1}^m d_{ij} \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) g_j(x_j(n - \ell)) + I_i \right] \right\} \end{aligned} \quad (7)$$

for $i = \overline{1, m}$, $n \geq n_0$, $n \neq n_k$, where $\psi_i(h) = \frac{1 - e^{-\beta_i h}}{\beta_i}$ denotes the associated denominator function. Observe that $0 < \psi_i(h) < \frac{1}{\beta_i}$ for $0 < h < \theta$ and $\psi_i(h) \approx h + O(h^2)$ for small $h > 0$.

The analogue (7) is supplemented with an initial vector sequence $\phi(\ell) = (\phi_1(\ell), \phi_2(\ell), \dots, \phi_m(\ell))^T$ for $\ell = n_0, n_0 - 1, n_0 - 2, \dots$ and is subject to impulsive state displacements characterized by the map

$$x_i(n_k^+) = x_i(n_k^-) + r_{ik}(x_i(n_k^-)), \quad i = \overline{1, m}, \quad k \in \mathbb{N}. \quad (8)$$

The iterations involved in (7) and (8) are described as follows: The notations n_k^+ and n_k^- denote the same integer n_k for which the values $x_i(n_k^-)$ generated by system (7) at time $n = n_k - 1$ are mapped impulsively by (8) to give the values $x_i(n_k^+)$. The mapped values $x_i(n_k^+)$ together with the past values $x_i(\ell)$ for $\ell = n_k - 1, n_k - 2, \dots$ are then supplied back to system (7) as initial values required for the next successive iterations of $x_i(n+1)$ for $n = n_k, n_k + 1, \dots, n_{k+1} - 1$. The existence of a unique solution $x(n) = (x_1(n), x_2(n), \dots, x_m(n))^T$ of the impulsive analogue (7), (8) for $n > n_0$ is therefore justified.

The impulsive map (8) was introduced in [22], wherein the impulse functions r_{ik} were defined linearly by $r_{ik}(x_i(n_k^-)) = -\delta_{ik}(x_i(n_k^-) - x_i^*)$ for $i = \overline{1, m}$, $k \in \mathbb{N}$ with δ_{ik} denoting real numbers. An iteration scheme similar to (7), (8) was

introduced in [8]. This map, for a given $h \in (0, \theta)$, differs from the impulsive state displacements described by difference equations of the form

$$x_i(n_k + 1) = x_i(n_k) + r_{ik}(x_i(n_k)), \quad i = \overline{1, m}, \quad k \in \mathbb{N},$$

considered in [1, 7, 11]. However, in the limit $h \rightarrow 0$, both characterizations operate in a similar manner in accordance with the continuous-time characterization (1) at the impulse moment $t = t_k$.

4. Exponential stability of equilibria

One can verify that an equilibrium point $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$ of the impulsive analogue (7), (8) satisfies the same algebraic system (2) and (3) under the assumptions \mathbf{A}_1 – \mathbf{A}_4 , \mathbf{A}'_5 for any given value $h \in (0, \theta)$. To prove the global exponential stability of the point x^* we will need the following definition and lemmas.

DEFINITION 1 ([33]). A real matrix $A = (a_{ij})_{m \times m}$ is said to be an M -matrix if $a_{ij} \leq 0$ for $i, j = \overline{1, m}$, $i \neq j$ and all successive principle minors of A are positive.

LEMMA 1 ([3]). Let $A = (a_{ij})_{m \times m}$ be a real matrix with non-positive off-diagonal elements. Then A is an M -matrix if and only if one of the following conditions holds:

- (1) There exists a vector $\xi = (\xi_1, \xi_2, \dots, \xi_m)^T$ with $\xi_i > 0$ such that every component of $\xi^T A$ is positive—that is, $\sum_{i=1}^m \xi_i a_{ij} > 0$, $j = \overline{1, m}$.
- (2) There exists a vector $\xi = (\xi_1, \xi_2, \dots, \xi_m)^T$ with $\xi_i > 0$ such that every component of $A\xi$ is positive—that is, $\sum_{j=1}^m a_{ij} \xi_j > 0$, $i = \overline{1, m}$.

For more details about M -matrices the reader is referred to [9, 13].

LEMMA 2 ([10]). A locally invertible C^0 map $H: \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a homeomorphism of \mathbb{R}^m onto itself if and only if it is proper.

In fact, this assertion is due to H a d a m a r d [12]. A mapping is proper if the pre-image of every compact is compact. In the finite-dimensional case it suffices to show that $\|H(x)\| \rightarrow \infty$ as $\|x\| \rightarrow \infty$.

Our first task is to prove the existence and uniqueness of the solution x^* of the algebraic system (2).

THEOREM 1. Let $p \geq 1$ be a real number, the value $h \in (0, \theta)$ be fixed and the assumptions \mathbf{A}_1 – \mathbf{A}_4 , \mathbf{A}'_5 hold. Suppose the matrix

$$\Xi_0 = B_0 - \frac{p-1}{p}(C_0^* + D_0^*) - \frac{1}{p}(C_0 L^f + D_0 L^g) \tag{9}$$

is an M -matrix, where

$$\begin{aligned} B_0 &= \text{diag}(\underline{b}_1, \underline{b}_2, \dots, \underline{b}_m), \quad C_0 = (|c_{ij}|)_{m \times m}, \quad D_0 = (|d_{ij}| \kappa_{ij})_{m \times m}, \\ C_0^* &= \text{diag} \left(\sum_{j=1}^m |c_{1j}| L_j^f, \sum_{j=1}^m |c_{2j}| L_j^f, \dots, \sum_{j=1}^m |c_{mj}| L_j^f \right), \\ D_0^* &= \text{diag} \left(\sum_{j=1}^m |d_{1j}| \kappa_{1j} L_j^g, \sum_{j=1}^m |d_{2j}| \kappa_{2j} L_j^g, \dots, \sum_{j=1}^m |d_{mj}| \kappa_{mj} L_j^g \right), \\ L^f &= \text{diag} \left(L_1^f, L_2^f, \dots, L_m^f \right), \quad L^g = \text{diag} \left(L_1^g, L_2^g, \dots, L_m^g \right). \end{aligned}$$

Then the algebraic system (2) has a unique solution $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T$.

Idea of the proof. Define a mapping $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$ by

$$F(x) = (F_1(x), F_2(x), \dots, F_m(x))^T \quad \text{for } x \in \mathbb{R}^m,$$

where

$$F_i(x) = -b_i(x_i) + \sum_{j=1}^m c_{ij} f_j(x_j) + \sum_{j=1}^m d_{ij} \kappa_{ij} g_j(x_j) + I_i, \quad i = \overline{1, m}.$$

The space \mathbb{R}^m is endowed with the norm $\|x\|_p = (\sum_{i=1}^m |x_i|^p)^{1/p}$. Under the assumptions $\mathbf{A}_2, \mathbf{A}_3$, $F(x) \in C^0$. It is known that if $F(x) \in C^0$ is a homeomorphism of \mathbb{R}^m , then there is a unique point $x^* = (x_1^*, x_2^*, \dots, x_m^*)^T \in \mathbb{R}^m$ such that $F(x^*) = 0$, that is, $F_i(x^*) = 0$, $i = \overline{1, m}$.

To demonstrate the one-to-one property of F , we take arbitrary vectors $x, y \in \mathbb{R}^m$ and assume that $F(x) = F(y)$. From

$$\begin{aligned} b_i(x_i) - b_i(y_i) &= \sum_{j=1}^m c_{ij} (f_j(x_j) - f_j(y_j)) \\ &\quad + \sum_{j=1}^m d_{ij} \kappa_{ij} (g_j(x_j) - g_j(y_j)), \quad i = \overline{1, m}, \end{aligned}$$

one obtains

$$\underline{b}_i |x_i - y_i| \leq \sum_{j=1}^m |c_{ij}| L_j^f |x_j - y_j| + \sum_{j=1}^m |d_{ij}| \kappa_{ij} L_j^g |x_j - y_j|, \quad i = \overline{1, m},$$

under the given assumptions. Multiplying both sides of the last inequality by $|x_i - y_i|^{p-1}$ and applying the inequality

$$\eta_1^{p-1} \eta_2 \leq \frac{p-1}{p} \eta_1^p + \frac{1}{p} \eta_2^p, \quad \eta_1, \eta_2 \geq 0, \quad p \geq 1,$$

we derive

$$\begin{aligned}
 \underline{b}_i |x_i - y_i|^p &\leq \sum_{j=1}^m |c_{ij}| L_j^f |x_i - y_i|^{p-1} |x_j - y_j| \\
 &\quad + \sum_{j=1}^m |d_{ij}| \kappa_{ij} L_j^g |x_i - y_i|^{p-1} |x_j - y_j| \\
 &\leq \sum_{j=1}^m |c_{ij}| L_j^f \left(\frac{p-1}{p} |x_i - y_i|^p + \frac{1}{p} |x_j - y_j|^p \right) \\
 &\quad + \sum_{j=1}^m |d_{ij}| \kappa_{ij} L_j^g \left(\frac{p-1}{p} |x_i - y_i|^p + \frac{1}{p} |x_j - y_j|^p \right), \quad i = \overline{1, m},
 \end{aligned}$$

which can be expressed as

$$\Xi_0(|x_1 - y_1|^p, |x_2 - y_2|^p, \dots, |x_m - y_m|^p)^T \leq 0.$$

The assertion $x_i = y_i$, $i = \overline{1, m}$, follows by virtue of Ξ_0 being an M -matrix. Thus $F(x) = F(y)$ implies $x = y$.

Finally, we show that $\|F(x)\|_p \rightarrow \infty$ as $\|x\|_p \rightarrow \infty$. According to Lemma 2, $F(x) \in C^0$ is a homeomorphism of \mathbb{R}^m . Thus there is a unique point $x^* \in \mathbb{R}^m$ such that $F(x^*) = 0$. The point represents a unique solution of the algebraic system (2). \square

The next task is to investigate the global exponential stability characteristics of the unique equilibrium point x^* of the impulse analogue (7), (8) for a fixed time-step $h \in (0, \theta)$. Upon introducing the translations

$$u_i(n) = x_i(n) - x_i^*, \quad \varphi_i(\ell) = \phi_i(\ell) - x_i^*, \quad \tilde{a}_i(u_i(n)) = a_i(u_i(n) + x_i^*),$$

$$\tilde{b}_i(u_i(n)) = b_i(u_i(n) + x_i^*) - b_i(x_i^*), \quad \tilde{f}_j(u_j(n)) = f_j(u_j(n) + x_j^*) - f_j(x_j^*),$$

$$\tilde{g}_j(u_j(n)) = g_j(u_j(n) + x_j^*) - g_j(x_j^*), \quad \tilde{r}_{ik}(u_i(n_k^-)) = r_{ik}(u_i(n_k^-) + x_i^*) + u_i(n_k^-),$$

one obtains

$$u_i(n+1) = e^{-\beta_i h} u_i(n) \tag{10}$$

$$\begin{aligned}
 &+ \psi_i(h) \left\{ \beta_i u_i(n) + \tilde{a}_i(u_i(n)) \left[-\tilde{b}_i(x_i(n)) + \sum_{j=1}^m c_{ij} \tilde{f}_j(u_j(n - \sigma_{ij})) \right. \right. \\
 &\quad \left. \left. + \sum_{j=1}^m d_{ij} \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) \tilde{g}_j(x_j(n - \ell)) \right] \right\}, \quad i = \overline{1, m}, \quad n \geq n_0, \quad n \neq n_k,
 \end{aligned}$$

$$u_i(n_k^+) = \tilde{r}_{ik}(u_i(n_k^-)), \quad i = \overline{1, m}, \quad k \in \mathbb{N}.$$

This system inherits the assumptions \mathbf{A}_1 – \mathbf{A}_4 , \mathbf{A}'_5 given before. In particular,

$$\tilde{r}_{ik}(0) = 0 \quad \text{for } i = \overline{1, m}, \quad k \in \mathbb{N}$$

and

$$|\tilde{r}_{ik}(u)| \leq (1 + \gamma_{ik})|u| \quad \text{for } i = \overline{1, m}, \quad k \in \mathbb{N}, \quad u \in \mathbb{R}, \quad (11)$$

where γ_{ik} denote positive real numbers.

Due to the equivalence between the systems (10) and (7), (8), it suffices to examine the exponential stability characteristics of the trivial equilibrium point $u^* = 0$ of the impulsive analogue (10). The main result is given by the following theorem.

THEOREM 2. *Let $p \geq 1$ be a real number, the value $h \in (0, \theta)$ be fixed and the assumptions \mathbf{A}_1 – \mathbf{A}_4 , \mathbf{A}'_5 hold. Suppose the matrix*

$$\Xi_1 = B_1 - \frac{p-1}{p}(C_1^* + D_1^*) - \frac{1}{p}(C_1 L^f + D_1 L^g) \quad (12)$$

is an M -matrix, where

$$\begin{aligned} B_1 &= \text{diag}(\underline{a}_1 \underline{b}_1, \underline{a}_2 \underline{b}_2, \dots, \underline{a}_m \underline{b}_m), \quad C_1 = (\bar{a}_i |c_{ij}|)_{m \times m}, \quad D_1 = (\bar{a}_i |d_{ij}| \kappa_{ij})_{m \times m}, \\ C_1^* &= \text{diag} \left(\sum_{j=1}^m \bar{a}_1 |c_{1j}| L_j^f, \sum_{j=1}^m \bar{a}_2 |c_{2j}| L_j^f, \dots, \sum_{j=1}^m \bar{a}_m |c_{mj}| L_j^f \right), \\ D_1^* &= \text{diag} \left(\sum_{j=1}^m \bar{a}_1 |d_{1j}| \kappa_{1j} L_j^g, \sum_{j=1}^m \bar{a}_2 |d_{2j}| \kappa_{2j} L_j^g, \dots, \sum_{j=1}^m \bar{a}_m |d_{mj}| \kappa_{mj} L_j^g \right). \end{aligned}$$

Suppose, further, that there exist positive numbers $\Lambda > 1$ and μ satisfying $\frac{\ln \Lambda}{\theta} < \mu < \nu$ for which

$$1 + \gamma_{ik} \leq \Lambda \quad \text{for } i = \overline{1, m}, \quad k \in \mathbb{N}. \quad (13)$$

Then the impulsive analogue (10) is globally exponentially stable with a Lyapunov exponent $\mu - \frac{\ln \Lambda}{\theta}$, namely,

$$\left[\sum_{i=1}^m |u_i(n)|^p \right]^{1/p} \leq \Omega e^{-(\mu - \frac{\ln \Lambda}{\theta})h(n-n_0)} \left[\sum_{i=1}^m \sup_{\ell \leq n_0} |\varphi_i(\ell)|^p \right]^{1/p} \quad (14)$$

for $n \geq n_0$, where $\Omega \geq 1$ denotes a constant.

Sketch of the proof. The property of the M -matrix Ξ_1 lends itself towards ascertaining that Ξ_0 is an M -matrix. Thus the existence and uniqueness of the equilibrium point x^* is assured by Theorem 1.

On applying the given assumptions to (10),

$$\begin{aligned}
 |u_i(n+1)| &\leq e^{-\beta_i h} |u_i(n)| \\
 &+ \psi_i(h) \left\{ \sum_{j=1}^m \bar{a}_i |c_{ij}| L_j^f |u_j(n - n_{ij})| \right. \\
 &\left. + \sum_{j=1}^m \bar{a}_i |d_{ij}| \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) L_j^g |u_j(n - \ell)| \right\}, \quad i = \overline{1, m}, \quad n \geq n_0, \quad n \neq n_k.
 \end{aligned} \tag{15}$$

Recall that $\beta_i = \underline{a}_i \underline{b}_i > 0$ and $\psi_i(h) = \frac{1 - e^{-\beta_i h}}{\beta_i} > 0$. From the M -matrix Ξ_1 and the property (1) of Lemma 1 one has that there is a positive vector $\xi = (\xi_1, \xi_2, \dots, \xi_m)^T$ for which

$$\begin{aligned}
 \xi_i \underline{a}_i \underline{b}_i - \frac{p-1}{p} \sum_{j=1}^m \xi_j \bar{a}_i |c_{ij}| L_j^f - \frac{p-1}{p} \sum_{j=1}^m \xi_j \bar{a}_i |d_{ij}| \kappa_{ij} L_j^g \\
 - \frac{1}{p} \sum_{j=1}^m \xi_j \bar{a}_j |c_{ji}| L_i^f - \frac{1}{p} \sum_{j=1}^m \xi_j \bar{a}_j |d_{ji}| \kappa_{ji} L_i^g > 0, \quad i = \overline{1, m}.
 \end{aligned}$$

Let us introduce a perturbation $\mu = \mu(h)$ for a fixed value $h \in (0, \theta)$ such that $0 < \mu < \min \{ \min_{i=\overline{1, m}} \beta_i, \nu \}$ and

$$\begin{aligned}
 \frac{e^{(\mu - \beta_i)h} - 1}{\psi_i(h)} + \frac{p-1}{p} \sum_{j=1}^m \bar{a}_i |c_{ij}| L_j^f e^{\mu(\sigma_{ij}+1)h} \\
 + \frac{p-1}{p} \sum_{j=1}^m \bar{a}_i |d_{ij}| L_j^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} \\
 + \frac{1}{p} \sum_{j=1}^m \frac{\xi_j}{\xi_i} \bar{a}_j |c_{ji}| L_i^f e^{\mu(\sigma_{ji}+1)h} \\
 + \frac{1}{p} \sum_{j=1}^m \frac{\xi_j}{\xi_i} \bar{a}_j |d_{ji}| L_i^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ji}(\ell) e^{\mu(\ell+1)h} \leq 0, \quad i = \overline{1, m}.
 \end{aligned} \tag{16}$$

Further, let

$$X_i(n) = e^{\mu(n-n_0)h} |u_i(n)| \quad \text{for } i = \overline{1, m}, \quad n \in \mathbb{Z}, \tag{17}$$

into the system (15). We obtain

$$\begin{aligned}
 X_i(n+1) &\leq e^{(\mu-\beta_i)h} X_i(n) \\
 &+ \psi_i(h) \left\{ \sum_{j=1}^m \bar{a}_i |c_{ij}| L_j^f e^{\mu(\sigma_{ij}+1)h} X_j(n-\sigma_{ij}) \right. \\
 &\left. + \sum_{j=1}^m \bar{a}_i |d_{ij}| \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) L_j^g e^{\mu(\ell+1)h} X_j(n-\ell) \right\}, \\
 & \qquad \qquad \qquad i = \overline{1, m}, \quad n \geq n_0, \quad n \neq n_k,
 \end{aligned}$$

which can be rearranged as

$$\begin{aligned}
 \frac{\Delta X_i(n)}{\psi_i(h)} &\leq \frac{e^{(\mu-\beta_i)h} - 1}{\psi_i(h)} X_i(n) \\
 &+ \sum_{j=1}^m \bar{a}_i |c_{ij}| L_j^f e^{\mu(\sigma_{ij}+1)h} X_j(n-\sigma_{ij}) \\
 &+ \sum_{j=1}^m \bar{a}_i |d_{ij}| \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) L_j^g e^{\mu(\ell+1)h} X_j(n-\ell), \\
 & \qquad \qquad \qquad i = \overline{1, m}, \quad n \geq n_0, \quad n \neq n_k.
 \end{aligned} \tag{18}$$

Define a Lyapunov sequence by

$$\begin{aligned}
 V(n) &= \sum_{i=1}^m \xi_i \psi_i^{-1}(h) X_i^p(n) \\
 &+ \sum_{i=1}^m \xi_i \sum_{j=1}^m \bar{a}_i |c_{ij}| L_j^f e^{\mu(\sigma_{ij}+1)h} \sum_{\ell=n-\sigma_{ij}}^{n-1} X_j^p(\ell) \\
 &+ \sum_{i=1}^m \xi_i \sum_{j=1}^m \bar{a}_i |d_{ij}| L_j^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} \sum_{r=n-\ell}^{n-1} X_j^p(r), \quad n \in \mathbb{Z}.
 \end{aligned} \tag{19}$$

One observes that the value

$$\begin{aligned}
 V(n_0) &\leq \max_{i=1, m} \left\{ \xi_i \psi_i^{-1}(h) + \sum_{j=1}^m \xi_j \bar{a}_j |c_{ji}| L_i^f e^{\mu(\sigma_{ji}+1)h} \sigma_{ji} \right. \\
 &\quad \left. + \sum_{j=1}^m \xi_j \bar{a}_j |d_{ji}| L_i^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ji}(\ell) e^{\mu(\ell+1)h} \ell \right\} \sum_{i=1}^m \sup_{r \leq n_0} X_i^p(r)
 \end{aligned}$$

is finite since $\sum_{\ell=1}^{\infty} \mathcal{K}_{ji}(\ell) e^{\mu(\ell+1)h} \ell < \infty$ by virtue of the assumption \mathbf{A}'_5 in which $0 < \mu < \nu$ for a fixed $h \in (0, \theta)$ and $\sup_{r \leq n_0} X_i^p(r) < \infty$ due to the boundedness of the initial sequence $\varphi(r) = \phi(r) - x^*$ for $r = n_0, n_0 - 1, n_0 - 2, \dots$

Now we estimate the forward difference $\Delta V(n) = V(n+1) - V(n)$ along the solutions of inequality (18), thus

$$\begin{aligned}
 \Delta V(n) &= \sum_{i=1}^m \xi_i \psi_i^{-1}(h) \Delta X_i^p(n) \\
 &+ \sum_{i=1}^m \xi_i \sum_{j=1}^m \bar{a}_i |c_{ij}| L_j^f e^{\mu(\sigma_{ij}+1)h} (X_j^p(n) - X_j^p(n - \sigma_{ij})) \\
 &+ \sum_{i=1}^m \xi_i \sum_{j=1}^m \bar{a}_i |d_{ij}| L_j^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} (X_j^p(n) - X_j^p(n - \ell)) \\
 &\leq \sum_{i=1}^m \xi_i p X_i^{p-1}(n) \left\{ \frac{e^{(\mu-\beta_i)h} - 1}{\psi_i(h)} X_i(n) + \sum_{j=1}^m \bar{a}_i |c_{ij}| L_j^f e^{\mu(\sigma_{ij}+1)h} X_j(n - \sigma_{ij}) \right. \\
 &\quad \left. + \sum_{j=1}^m \bar{a}_i |d_{ij}| L_j^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} X_j(n - \ell) \right\} \\
 &+ \sum_{i=1}^m \xi_i \sum_{j=1}^m \bar{a}_i |c_{ij}| L_j^f e^{\mu(\sigma_{ij}+1)h} (X_j^p(n) - X_j^p(n - \sigma_{ij})) \\
 &+ \sum_{i=1}^m \xi_i \sum_{j=1}^m \bar{a}_i |d_{ij}| L_j^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} (X_j^p(n) - X_j^p(n - \ell)) \\
 &\leq \sum_{i=1}^m \xi_i p \left\{ \frac{e^{(\mu-\beta_i)h} - 1}{\psi_i(h)} X_i^p(n) \right. \\
 &\quad + \sum_{j=1}^m \bar{a}_i |c_{ij}| L_j^f e^{\mu(\sigma_{ij}+1)h} \left(\frac{p-1}{p} X_i^p(n) + \frac{1}{p} X_j^p(n - \sigma_{ij}) \right) \\
 &\quad + \sum_{j=1}^m \bar{a}_i |d_{ij}| L_j^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} \left(\frac{p-1}{p} X_i^p(n) + \frac{1}{p} X_j^p(n - \ell) \right) \\
 &\quad + \sum_{j=1}^m \bar{a}_i |c_{ij}| L_j^f e^{\mu(\sigma_{ij}+1)h} \left(\frac{1}{p} X_j^p(n) - \frac{1}{p} X_j^p(n - \sigma_{ij}) \right) \\
 &\quad \left. + \sum_{j=1}^m \bar{a}_i |d_{ij}| L_j^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} \left(\frac{1}{p} X_j^p(n) - \frac{1}{p} X_j^p(n - \ell) \right) \right\} \\
 &= \sum_{i=1}^m \xi_i p \left\{ \frac{e^{(\mu-\beta_i)h} - 1}{\psi_i(h)} + \frac{p-1}{p} \sum_{j=1}^m \bar{a}_i |c_{ij}| L_j^f e^{\mu(\sigma_{ij}+1)h} \right. \\
 &\quad + \frac{p-1}{p} \sum_{j=1}^m \bar{a}_i |d_{ij}| L_j^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ij}(\ell) e^{\mu(\ell+1)h} + \frac{1}{p} \sum_{j=1}^m \frac{\xi_j}{\xi_i} \bar{a}_j |c_{ji}| L_i^f e^{\mu(\sigma_{ji}+1)h} \\
 &\quad \left. + \frac{1}{p} \sum_{j=1}^m \frac{\xi_j}{\xi_i} \bar{a}_j |d_{ji}| L_i^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ji}(\ell) e^{\mu(\ell+1)h} \right\} X_i^p(n)
 \end{aligned}$$

for $n \geq n_0$, $n \neq n_k$. By virtue of the condition (16) we find that $\Delta V(n) \leq 0$ for $n \geq n_0$, $n \neq n_k$. This means that

$$V(n) \leq V(n_{k-1}^+) \quad \text{for } n_{k-1} \leq n < n_k, \quad k \in \mathbb{N}. \quad (20)$$

Next,

$$X_i^p(n_{k-1}^+) \leq (1 + \gamma_{i(k-1)})^p X_i^p(n_{k-1}^-) \leq \Lambda^p X_i^p(n_{k-1}^-)$$

for $k = 1, 2, 3, \dots$, which implies

$$V(n_{k-1}^+) \leq \Lambda^p V(n_{k-1}^-) \quad \text{for } k \in \mathbb{N}. \quad (21)$$

From the statements (20) and (21) it follows that

$$V(n) \leq \Lambda^{(k-1)p} V(n_0) \quad \text{for } n_0 \leq n < n_k, \quad k \in \mathbb{N}. \quad (22)$$

Since

$$h(n - n_0) \geq (k - 1)\theta \quad \text{for } n_0 \leq n < n_k, \quad k \in \mathbb{N},$$

from (22) we derive

$$V(n) \leq \Lambda^{p \frac{h(n-n_0)}{\theta}} V(n_0) = e^{p \frac{\ln \Lambda}{\theta} h(n-n_0)} V(n_0) \quad \text{for } n \geq n_0.$$

This estimate together with (17) and (19) leads to

$$\sum_{i=1}^m \xi_i \psi_i^{-1} e^{\mu p h(n-n_0)} |u_i(n)|^p \leq V(n) \leq e^{p \frac{\ln \Lambda}{\theta} h(n-n_0)} V(n_0)$$

for $n \geq n_0$, from which

$$\sum_{i=1}^m |u_i(n)|^p \leq \Omega_1 e^{-p(\mu - \frac{\ln \Lambda}{\theta})h(n-n_0)} \sum_{i=1}^m \sup_{\ell \leq n_0} |\varphi_i(\ell)|^p$$

for $n \geq n_0$, where the constant

$$\Omega_1 = \max_{i=1, m} \left\{ \xi_i \psi_i^{-1}(h) + \sum_{j=1}^m \xi_j \bar{a}_j |c_{ji}| L_i^f e^{\mu(\sigma_{ji}+1)h} \sigma_{ji} + \sum_{j=1}^m \xi_j \bar{a}_j |d_{ji}| L_i^g \sum_{\ell=1}^{\infty} \mathcal{K}_{ji}(\ell) e^{\mu(\ell+1)h} \ell \right\} / \min_{i=1, m} \{ \xi_i \psi_i^{-1}(h) \}$$

satisfies $1 \leq \Omega_1 < \infty$. The statement (14) will follow subsequently and this completes the proof. \square

COROLLARY 1. *Suppose the assumptions and conditions in Theorem 2 are satisfied with (12) replaced by*

$$\begin{aligned} \underline{a}_i \underline{b}_i - \frac{p-1}{p} \sum_{j=1}^m \bar{a}_i |c_{ij}| L_j^f - \frac{p-1}{p} \sum_{j=1}^m \bar{a}_i |d_{ij}| \kappa_{ij} L_j^g \\ - \frac{1}{p} \sum_{j=1}^m \bar{a}_j |c_{ji}| L_i^f - \frac{1}{p} \sum_{j=1}^m \bar{a}_j |d_{ji}| \kappa_{ji} L_i^g > 0, \quad i = \overline{1, m}. \end{aligned} \quad (23)$$

Then the impulsive analogue (10) is globally exponentially stable in the sense of (14).

In fact, the inequalities (23) imply that Ξ_1 defined in (12) becomes an M -matrix.

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