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Almost Periodic Solutions of Neutral Impulsive Systems with Periodic Time-Dependent Perturbed Delays

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Abstract: A neutral impulsive system with a small delay of the argument of the derivative and another delay which differs from a constant by a periodic perturbation of a small amplitude is considered. If the corresponding system with constant delay has an isolated ω-periodic solution and the period of the delay is not rationally dependent on ω, then under a nondegeneracy assumption it is proved that in any sufficiently small neighbourhood of this orbit the perturbed system has a unique almost periodic solution.

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1 Introduction

In the mathematical simulation of the evolution of real processes in physics, chemistry, population dynamics, radio engineering etc. which are subject to disturbances of negligible duration with respect to the total duration of the process, it is often convenient to assume that the disturbances are "momentary", in the form of impulses. This leads to the investigation of differential equations and systems with discontinuous trajectories, or with impulse effect, called for the sake of brevity impulsive differential equations and systems.

Impulsive differential equations with delay describe models of real processes and phenomena where both dependence on the past and momentary disturbances are observed. For instance, the size of a given population may be normally described by a delay differential equation and, at certain moments, the number of individuals can be abruptly changed. The interaction of the impulsive perturbation and the delay makes difficult the qualitative investigation of such equations. In particular, the solutions are not smooth at the moments of impulse effect shifted by the delay [8].

Suppose that there are finitely many argument deviations whose dependence on t is known. Then the general form of the functional differential equation is

$$x^{(m)}(t) = f(t, x^{(m_1)}(t - h_1(t)), \ldots, x^{(m_k)}(t - h_k(t))) \tag{1}$$

where $x(t) \in \mathbb{R}^n$: real Euclidean space of n-dimensional column vectors with norm $|\cdot|$ and $m_i \geq 0,\ h_i \geq 0$ for all i.

i) Equation (1) is called *a functional differential equation of retarded type, or retarded functional differential equation* if $\max\{m_1, m_2, \ldots, m_k\} < m$.

ii) Equation (1) is called *a functional differential equation of neutral type* if $\max\{m_1, m_2, \ldots, m_k\} = m$.

iii) Equation (1) is called *a functional differential equation of advanced type* if $\max\{m_1, m_2, \ldots, m_k\} > m$.

A classical problem of the qualitative theory of differential equations is the existence of periodic (or almost periodic) solutions. Numerous references on this matter concerning differential equations with delay and impulsive differential equations can be found in [5]. A traditional approach to this problem is the investigation of the linearized system (also called *system in variations*) with respect to a periodic solution of the unperturbed system satisfying certain nondegeneracy assumptions.

In [5] it is proved, by using the implicit function theorem, that for a periodic impulsive system with a small delay, if the corresponding system without delay has an isolated ω-periodic solution, then in any sufficiently small neighbourhood of this orbit, the system considered also has a unique ω-periodic solution. In an earlier version [7] of this paper, this result was proved under considerably more restrictive assumptions with the use of the contraction mapping principle (see also [4], §8). Moreover, this result was extended to the case of a neutral impulsive system with a small delay [13].

In [9, 10] a periodic impulsive system with a small delay is considered, such that the corresponding system without delay is linear and has an r-parametric family of ω-

periodic solutions. In the so called critical cases of the first and second order conditions are obtained for the existence of ω-periodic solutions of the initial system.

In the papers [1, 2, 11], the problem of existence of periodic (both in the noncritical and critical cases) and almost periodic solutions is studied in the presence of a delay which differs from a constant by a small amplitude periodic perturbation. The introduction of such delays is justified in the abstract [16] as follows:

> "There are many reasons for incorporating delays into biological and ecological models, and often a fixed delay is best suited, or a good approximation. The strength of these fixed delays may however fluctuate in time, through e.g. seasonal effects or photoperiod. We would e.g. expect the regeneration rate of resources to be slower in winter than summer. In addition, the effect of small perturbations in model parameters (including delay) can give information on the robustness of our model, since model parameters invariably need to be estimated.
>
> We examine some simple population models that incorporate a time delay which is not a constant but is instead an explicit periodic function of time ...".

In the present paper we consider a neutral impulsive system with a small delay of the argument of the derivative and another delay which differs from a constant by a periodic perturbation of a small amplitude . If the corresponding system with constant delay has an isolated ω-periodic solution and the period of the delay is not rationally dependent on ω, then under a nondegeneracy assumption we shall prove that in any sufficiently small neighbourhood of this orbit the perturbed system has a unique almost periodic solution. The case when the period of the delay is ω (or is a rational multiple of ω) will be considered elsewhere.

2 Statement of the problem. Main result

Throughout this paper we study a neutral system with impulses at fixed moments and a small delay of the argument of the derivative. Another delay fluctuating around a constant value which can be assumed to be 1 without loss of generality:

$$\begin{cases} \dot{x}(t) = D(t)\dot{x}(t-h) + f(t, x(t), x(t-h), x(t-1-h\varphi(t))), \ t \neq t_i, \\ \Delta x(t_i) = I_i(x(t_i), x(t_i - h)), \ i \in \mathbb{Z}, \end{cases} \tag{2}$$

where $x \in \Omega \subset \mathbb{R}^n$, $D : \mathbb{R} \to \mathbb{R}^{n\times n}$, $f : \mathbb{R} \times \Omega \times \Omega \times \Omega \to \mathbb{R}^n$, Ω is a domain in $\mathbb{R}^n$; $\varphi : \mathbb{R} \to [-1, 1]$; $\mathbb{Z}$ is the set of all integers; $\Delta x(t_i) = x(t_i+0)-x(t_i-0)$ are the impulses at moments t_i and $\{t_i\}_{i\in\mathbb{Z}}$ is a strictly increasing sequence such that $\lim\limits_{i\to\pm\infty} t_i = \pm\infty$; $I_i : \Omega \times \Omega \to \mathbb{R}^n$ $(i \in \mathbb{Z})$, h and $1 + h\varphi(t)$ are the delays, $h \in [0, h_0)$ is a small parameter; h_0 will be specified below.

As usual in the theory of the impulsive differential equations [4, 14], at the points of discontinuity t_i of the solution $x(t)$ we assume that $x(t_i) \equiv x(t_i - 0)$. It is clear that, in general, the derivatives $\dot{x}(t_i + kh)$, $k \in \mathbb{Z}$, do not exist. On the other hand, there do exist the limits $\dot{x}(t_i + kh \pm 0)$. According to the above convention, we assume $\dot{x}(t_i + kh) \equiv \dot{x}(t_i + kh - 0)$.

Similarly, the derivative $\dot{x}$ does not exist at the other points of discontinuity of the right-hand side $f(t, x(t), x(t-h), x(t-1-h\varphi(t)))$, *i.e.*, at points t which are solutions of the equations

$$t - 1 - h\varphi(t) = t_i, \tag{3}$$

$i \in \mathbb{Z}$. We require the continuity of the solution $x(t)$ at such points if they are distinct from the moments of impulse effect t_i.

For the sake of brevity we shall use the notation:

$$x_i = x(t_i), \quad \bar{x}(t) = x(t-h), \quad \tilde{x}(t) = x(t-1), \quad y^h(t) = x(t-1-h\varphi(t))$$

(thus, for instance, $y_i^0 = x(t_i - 1) = \tilde{x}_i$).

We let $|x|$ denote the Euclidean norm of a vector $x \in \mathbb{R}^n$, and for an $(n \times n)$–matrix A we define the associated norm

$$|A| \;=\; \sup\,\{|Ax|/|x|; \quad x \in \mathbb{R}^n \setminus 0\}.$$

In the sequel we require the fulfillment of the following assumptions:

A1. The function $f(t, x, \bar{x}, y)$ is continuous (or piecewise continuous, with discontinuities of the first kind at the points t_i) and ω-periodic with respect to t, twice continuously differentiable with respect to $x, \bar{x}, y \in \Omega$, with second derivatives that are locally Lipschitz continuous with respect to $x, \bar{x}, y$.

A2. The matrix $D(t)$ is ω-periodic, $\sup\limits_{t\in[0,\omega]} |D(t)| = \eta < 1$, its first derivative is continuous (or piecewise continuous, with discontinuities of the first kind at the points t_i), and its second derivative is bounded on each interval of continuity.

A3. The functions $I_i(x, \bar{x})$, $i \in \mathbb{Z}$, are twice continuously differentiable with respect to $x, \bar{x} \in \Omega$, with second derivatives that are locally Lipschitz continuous with respect to $x, \bar{x}$.

A4. There exists a positive integer m such that $t_{i+m} = t_i + \omega$, $I_{i+m}(x, \bar{x}) = I_i(x, \bar{x})$ for $i \in \mathbb{Z}$ and $x, \bar{x} \in \Omega$.

A5. The function $\varphi(t)$ is ω_1–periodic, where ω_1/ω is irrational, and Lipschitz continuous:

$$|\varphi(t') - \varphi(t'')| \;\le\; K|t' - t''|, \quad t', t'' \in \mathbb{R}.$$

We may note that the invertibility of the matrix $E - D(t)$ (E is the unit matrix) follows from the inequality $\eta < 1$ (condition A2). Moreover, $\sup\limits_{t\in[0,\omega]} |(E - D(t))^{-1}| \le (1-\eta)^{-1}$.

If $h_0 \le \min\,\{1,\, 1/K\}$, then for $h \in (0, h_0)$ equation (3) has a unique solution $t_i(h)$ for each $i \in \mathbb{Z}$. It obviously satisfies

$$|t_i(h) - t_i - 1| \;\le\; h, \quad t_i(0) = t_i + 1.$$

It is natural to assume that the period ω is distinct from the unperturbed delay 1. For the sake of definiteness we assume that $\omega > 1$ and $t_i \ne 0\ \forall i \in \mathbb{Z}$.

For $h = 0$, we use (2) to obtain the so called *generating system*

$$\begin{cases} \dot{x}(t) = (E - D(t))^{-1} f(t, x(t), x(t), x(t-1)),\ t \neq t_i, \\ \Delta x(t_i) = I_i(x_i, x_i),\ i \in \mathbb{Z}, \end{cases} \tag{4}$$

and suppose that

A6. The generating system (4) has an ω-periodic solution $\psi(t)$ such that $\psi(t) \in \Omega$ for all $t \in \mathbb{R}$, and

A7. $\left.\dfrac{\partial}{\partial y} f(t, \psi(t), \psi(t), y)\right|_{y=\psi(t-1)} = 0$, the zero matrix of dimension $(n \times n)$.

Now define the linearized system with respect to $\psi(t)$:

$$\begin{cases} \dot{z}(t) &= (E - D(t))^{-1} A(t) z(t),\ t \neq t_i, \\ \Delta z(t_i) &= B_i z_i,\ i \in \mathbb{Z}, \end{cases} \tag{5}$$

where

$$A(t) = \left.\frac{\partial}{\partial x} f(t, x, x, \psi(t-1))\right|_{x=\psi(t)}, \quad \text{and} \quad B_i = \left.\frac{\partial}{\partial x} I_i(x, x)\right|_{x=\psi_i}.$$

Let the $(n \times n)$–matrix $X(t, s)$ be the Cauchy matrix of (5), and let $X(t) = X(t, 0)$ be its fundamental solution [15]. Denote

$$\Lambda = \frac{1}{\omega} \ln X(\omega), \qquad \Phi(t) = X(t) e^{-\Lambda t}.$$

$\Phi(t)$ is an ω-periodic piecewise continuous nondegenerate matrix-valued function, with points of discontinuity of the first kind at $\{t_i\}_{i \in \mathbb{Z}}$. Now we make two additional assumptions:

A8. The matrices $E + B_i$, $i \in \mathbf{Z}$, are nonsingular.

A9. The matrix Λ has no eigenvalues with real part zero.

Together with (5) we consider the nonhomogeneous system

$$\begin{cases} \dot{z}(t) = (E - D(t))^{-1} \big(A(t) z(t) + f(t) \big),\ t \neq t_i, \\ \Delta z(t_i) = B_i z_i + a_i, \qquad i \in \mathbb{Z}, \end{cases} \tag{6}$$

where $f(t) \in AP_n\{t_i\}$, the space of all almost periodic functions with values in $\mathbb{R}^n$, which are piecewise continuous with points of discontinuity of the first kind at t_i, $i \in \mathbb{Z}$, while $a_i \in ap_n$, the space of all almost periodic sequences with values in $\mathbb{R}^n$ [15]. Under these assumptions system (6) has a unique almost periodic solution (see [15, Theorem 25.3]).

We give only those fragments of the proof that will be used henceforth.

Without loss of generality we may assume that $\Lambda = \operatorname{diag}(P, N)$, where P and N are square matrices of order k and $n - k$ respectively, such that

$$\operatorname{Re} \lambda_j(P) > 0, \quad j = \overline{1, k}, \qquad \operatorname{Re} \lambda_j(N) < 0, \quad j = \overline{k+1, n}.$$

Denote

$$G(t)=\begin{cases}-\operatorname{diag}\left(e^{Pt},0\right) & \text{for } \ t<0,\\ \\ \operatorname{diag}\left(0,e^{Nt}\right) & \text{for } \ t>0.\end{cases}$$

It can be shown that

$$\|G(t)\|\le Ce^{-\alpha|t|}, \tag{7}$$

where C and α are positive constants. Moreover,

$$z_0(t)=\int_{-\infty}^{\infty}\Phi(t)G(t-\tau)\Phi^{-1}(\tau)(E-D(\tau))^{-1}f(\tau)\,d\tau+\sum_{i\in\mathbf{Z}}\Phi(t)G(t-t_i)\Phi^{-1}(t_i)a_i \tag{8}$$

is the unique almost periodic solution of (6). We will also need to estimate

$$\int_{-\infty}^{\infty}\|G(t-\tau)\|\,d\tau \qquad \text{and} \qquad \sum_{i\in\mathbf{Z}}\|G(t-t_i)\|$$

by using of (7). We have

$$\int_{-\infty}^{\infty}\|G(t-\tau)\|\,d\tau\le C\int_{-\infty}^{\infty}e^{-\alpha|t-\tau|}\,d\tau=2C\int_{0}^{\infty}e^{-\alpha\sigma}\,d\sigma=\frac{2C}{\alpha}. \tag{9}$$

Next we estimate

$$S(t)=\sum_{i\in\mathbb{Z}}e^{-\alpha|t-t_i|}$$

under the assumption that

$$\inf_{j\in\mathbb{Z}}(t_{j+1}-t_j)=\theta>0.$$

In our case $\theta=\min\limits_{0\le j\le m-1}(t_{j+1}-t_j)$. Without loss of generality we may assume that $t_0\le t<t_1$. Then

$$S(t)=\sum_{i=1}^{\infty}e^{-\alpha(t_i-t)}+\sum_{i=0}^{\infty}e^{-\alpha(t-t_{-i})}.$$

In the first sum

$$t_i-t\ge t_i-t_1=\sum_{j=1}^{i-1}(t_{j+1}-t_j)\ge(i-1)\theta,$$

and in the second one

$$t-t_{-i}\ge t_0-t_{-i}=\sum_{j=0}^{i-1}(t_{-j}-t_{-j-1})\ge i\theta.$$

So we have

$$S(t)\le\sum_{i=1}^{\infty}e^{-\alpha\theta(i-1)}+\sum_{i=0}^{\infty}e^{-\alpha\theta i}=\frac{2}{1-e^{-\alpha\theta}}$$

and

$$\sum_{i\in\mathbf{Z}}\|G(t-t_i)\|\le\frac{2C}{1-e^{-\alpha\theta}}. \tag{10}$$

Denote also

$$\mathcal{M} = \sup\left\{\|\Phi(t)\|\|\Phi^{-1}(\tau)\| : t,\tau\in[0,\omega]\right\}, \qquad \beta = \sup_{t\in\mathbb{R}}\sum_{i\in\mathbb{Z}}\|G(t-t_i)\|\,|B_i|.$$

In fact, the last supremum does not exceed

$$\frac{2C}{1-e^{-\alpha\theta}}\sup_{i\in\mathbb{Z}}|B_i|.$$

Our result in the present paper is the following

Theorem 2.1. Let conditions A1–A9 hold. If

$$\eta(3+2\beta\mathcal{M})<1, \tag{11}$$

then there exists a number $h_*>0$ such that for $h\in(0,h_*)$ system (2) has a unique almost periodic solution $x(t,h)$ depending continuously on h and $x(t,h)\to\psi(t)$ as $h\to 0$.

Remark 2.2. Condition A7 is of technical character. It enables us to apply Floquet's theory adapted for impulsive systems in [15]. Otherwise we would have to adapt the spectral decompositions given in [12] for impulsive systems and apply them to our case. Conditions of the form A1, A3 first appeared in [4], §8 and [7] even though they were not required by the method applied there. In the subsequent papers [1, 2, 5, 9, 11] and [10] (only in the critical case of first order) they were weakened to continuity of the first partial derivatives with respect to $x,\bar{x},y$. However, this is not possible for neutral systems when we use the contraction mapping principle as in [13] or the present paper. Still under such weaker conditions (and just piecewise continuity of $D(t)$ in A2) we can prove the existence of a not necessarily unique periodic (respectively almost periodic) solution by using Schauder's fixed point theorem (see [6]).

3 Proof of the main result

In system (2) we change the variables according to the formula

$$x=\psi(t)+z \tag{12}$$

and obtain the system

$$\begin{cases}\dot{z}(t) &= (E-D(t))^{-1}\big\{A(t)z(t)+Q(t,z(t),z(t-1))\\ &\quad +\delta f(t,x(t),x(t-h),y^h(t))-D(t)(\dot{x}(t)-\dot{x}(t-h))\big\},\ t\neq t_i,\\ \Delta z(t_i) &= B_iz_i+J_i(z_i)+\delta I_i(x_i,\bar{x}_i),\ i\in\mathbb{Z},\end{cases} \tag{13}$$

where

$$\begin{aligned} Q(t,z,\tilde{z}) &\equiv f(t,\psi(t)+z,\psi(t)+z,\psi(t-1)+\tilde{z})\\ &\quad -f(t,\psi(t),\psi(t),\psi(t-1)) - A(t)z,\\ J_i(z_i) &\equiv I_i(\psi_i+z_i,\psi_i+z_i)-I_i(\psi_i,\psi_i)-B_iz_i\end{aligned}$$

are nonlinearities inherent to the generating system (4) and therefore independent of the small parameter h, while

$$\delta f(t,x(t),\bar{x}(t),y^h(t)) \equiv f(t,x(t),\bar{x}(t),y^h(t)) - f(t,x(t),x(t),y^0(t)),$$
$$\delta I_i(x_i,\bar{x}_i) \equiv I_i(x_i,\bar{x}_i) - I_i(x_i,x_i)$$

are increments due to the presence of the small parameter.

We can formally consider (13) as a nonhomogeneous system of the form (6). Then its unique almost periodic solution $z(t)$ must satisfy an equality of the form (8) which in this case is the operator equation

$$z = \mathcal{U}_h z - \mathcal{V}_h z, \tag{14}$$

where

$$\begin{aligned}\mathcal{U}_h z(t) \equiv & \int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\Phi^{-1}(\tau)(E-D(\tau))^{-1}Q(\tau,z(\tau),z(\tau-1))\,d\tau \\ & + \int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\Phi^{-1}(\tau)(E-D(\tau))^{-1}\delta f(\tau,x(\tau),\bar{x}(\tau),y^h(\tau))\,d\tau \\ & + \sum_{i\in\mathbb{Z}} \Phi(t)G(t-t_i)\Phi^{-1}(t_i)J_i(z_i) + \sum_{i\in\mathbb{Z}} \Phi(t)G(t-t_i)\Phi^{-1}(t_i)\delta I_i(x_i,\bar{x}_i) \\ \equiv & \ \mathcal{I}_1 z(t) + \mathcal{I}_2 z(t) + \mathcal{S}_1 z(t) + \mathcal{S}_2 z(t),\end{aligned}$$

$$\mathcal{V}_h z(t) \equiv \int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\Phi^{-1}(\tau)(E-D(\tau))^{-1}D(\tau)\big(\dot{x}(\tau)-\dot{x}(\tau-h)\big)\,d\tau. \tag{15}$$

For the sake of brevity we still write x instead of $\psi(t)+z$ in $\delta f(t,x(t),\bar{x}(t),y^h(t))$, $\delta I_i(x_i,\bar{x}_i)$ as well as in $\mathcal{I}_2 z$, $\mathcal{S}_2 z$ and in $\mathcal{V}_h z$. Moreover, in §3.2 we will further transform the expressions $\mathcal{I}_2 z(t)$ and $\mathcal{S}_2 z(t)$ under the assumption that $x(t)$ is a solution of system (2). This will considerably simplify some estimates in §3.2 and §3.3. Such additional modification of the operator equation was used in [9, 10] for a periodic system with small constant delay and a linear generating system in the critical case (when the linearized system has nontrivial ω-periodic solutions).

An almost periodic solution $x(t) = x(t,h)$ of system (2) corresponds to a fixed point z of the operator $\mathcal{U}_h - \mathcal{V}_h$ in a suitable set of almost periodic functions. To this end we shall prove that $\mathcal{U}_h - \mathcal{V}_h$ maps a suitably chosen set into itself (§3.2) as a contraction (§3.3).

We first need to introduce some

3.1 Notation

We assume that the spaces $AP_n\{t_i\}$ and ap_n of almost periodic functions and almost periodic sequences respectively are equipped with the norms

$$\|w\| = \sup_{t\in\mathbb{R}} |w(t)|, \qquad \|\{a_i\}_{i\in\mathbb{Z}}\| = \sup_{i\in\mathbb{Z}} |a_i|.$$

There exists a constant μ_0 such that Ω contains a closed μ_0–neighbourhood Ω_1 of the periodic orbit $\{x = \psi(t);\ t \in \mathbb{R}\}$. Let us denote

$$M_0 = \max\Big\{\sup\big\{|f(t,x,\bar{x},y)| :\ t \in [0,\omega],\ x,\bar{x},y \in \Omega_1\big\},\\ \sup\big\{|I_i(x,\bar{x})| :\ i = \overline{1,m},\ x,\bar{x} \in \Omega_1\big\}\Big\},$$

$$M_1 = \max\Big\{\sup\big\{|\partial_x f(t,x,\bar{x},y)| :\ t \in [0,\omega],\ x,\bar{x},y \in \Omega_1\big\},\\ \sup\big\{|\partial_{\bar{x}} f(t,x,\bar{x},y)| :\ t \in [0,\omega],\ x,\bar{x},y \in \Omega_1\big\},\\ \sup\big\{|\partial_y f(t,x,\bar{x},y)| :\ t \in [0,\omega],\ x,\bar{x},y \in \Omega_1\big\},\\ \sup\big\{|\partial_x I_i(x,\bar{x})| :\ i = \overline{1,m},\ x,\bar{x} \in \Omega_1\big\},\\ \sup\big\{|\partial_{\bar{x}} I_i(x,\bar{x})| :\ i = \overline{1,m},\ x,y \in \Omega_1\big\}\Big\}$$

and, similarly, let M_2 be the maximum of the supremums of the matrices of the second derivatives of $f(t,x,\bar{x},y)$ with respect to $x,\bar{x},y$ for $t \in [0,\omega],\ x,\bar{x},y \in \Omega_1$ and of the second derivatives of $I_i(x,\bar{x})$ for $i = \overline{1,m},\ x,y \in \Omega_1$. We shall not explicitly denote the Lipschitz constants for the second derivatives of $f(t,x,\bar{x},y)$ and $I_i(x,\bar{x})$. Sometimes, for the sake of brevity, we shall use the Landau symbol $O(\mu^k)$ for a quantity whose module (norm) can be estimated by a constant times μ^k for μ small enough. The meaning of $O(h)$ is similar.

For $a,b \in \mathbb{R}$ denote

$$]a,b[= \begin{cases} (a,b) & \text{if } a < b, \\ (b,a) & \text{if } a > b, \\ \emptyset & \text{if } a = b. \end{cases}$$

We may note that

$$\tau \in]t_i(h), t_i + 1[\quad \Longleftrightarrow \quad t_i \in]\tau - 1, \tau - 1 - h\varphi(\tau)[.$$

Define the "bad" sets

$$\Delta_1^h = \bigcup_{i\in\mathbb{Z}} (t_i, t_i + h), \qquad \Delta_2^h = \bigcup_{i\in\mathbb{Z}}]t_i(h), t_i + 1[.$$

We further define the "good" set $\Delta_3^h = \mathbb{R} \setminus (\Delta_1^h \cup \Delta_2^h)$.

For the sake of convenience we assume that for $i = \overline{1,m} \quad t_i + 1 \neq t_j\ \forall j \in \mathbb{Z}$. Then for h small enough the "bad" set $\Delta_1^h \cup \Delta_2^h$ is a disjoint union of intervals.

Let $h_0 > 0$ be so small that all the above assumptions are valid for $h \in (0,h_0)$.

For $\mu \in (0,\mu_0]$ define a set of functions

$$\mathcal{T}_\mu \ = \ \{\ z \in AP_n :\quad \|z\| \le \mu\ \}.$$

We shall find a dependence between h and μ so that the operator $\mathcal{U}_h - \mathcal{V}_h$ in (14) maps the set $\mathcal{T}_\mu$ into itself as a contraction.

3.2 Invariance of the set $\mathcal{T}_\mu$ under the action of the operator $\mathcal{U}_h - \mathcal{V}_h$

Let $z \in \mathcal{T}_\mu$. We shall estimate $|\mathcal{U}_h z(t)|$ using the representation

$$\mathcal{U}_h z(t) \;=\; \mathcal{I}_1 z(t) + \mathcal{I}_2 z(t) + \mathcal{S}_1 z(t) + \mathcal{S}_2 z(t)$$

and system (2).

First we have

$$J_i(z_i) = \Big\{ \int_0^1 \left(\partial_x I_i(\psi_i + s z_i, \psi_i + s z_i) \;-\; \partial_x I_i(\psi_i, \psi_i)\right) ds \\ + \int_0^1 \left(\partial_{\bar{x}} I_i(\psi_i + s z_i, \psi_i + s z_i) \;-\; \partial_{\bar{x}} I_i(\psi_i, \psi_i)\right) ds \Big\} z_i,$$

thus

$$|J_i(z_i)| \le 2 \int_0^1 2 M_2 s |z_i| \, ds \cdot |z_i| = 2 M_2 |z_i|^2$$

and

$$|\mathcal{S}_1 z(t)| \le \sum_{i \in \mathbb{Z}} \|\Phi(t) G(t - t_i) \Phi^{-1}(t_i)\| \, \|J_i(z_i)\| \le 2 M_2 \mathcal{M} \sum_{i \in \mathbb{Z}} \|G(t - t_i)\| |z_i|^2 = O(\mu^2). \tag{16}$$

Similarly, we have

$$Q(\tau, z(\tau), z(\tau-1)) = \Big\{ \int_0^1 \big[\partial_x f(\tau, \psi(\tau) + s z(\tau), \psi(\tau) + s z(\tau), \tilde{\psi}(\tau) + s \tilde{z}(\tau)) \\ - \partial_x f(\tau, \psi(\tau), \psi(\tau), \tilde{\psi}(\tau))\big] ds \\ + \int_0^1 \big[\partial_{\bar{x}} f(\tau, \psi(\tau) + s z(\tau), \psi(\tau) + s z(\tau), \tilde{\psi}(\tau) + s \tilde{z}(\tau)) \\ - \partial_{\bar{x}} f(\tau, \psi(\tau), \psi(\tau), \tilde{\psi}(\tau))\big] ds \Big\} z(\tau) \\ + \int_0^1 \big[\partial_y f(\tau, \psi(\tau) + s z(\tau), \psi(\tau) + s z(\tau), \tilde{\psi}(\tau) + s \tilde{z}(\tau)) \\ - \partial_y f(\tau, \psi(\tau), \psi(\tau), \tilde{\psi}(\tau))\big] ds \cdot \tilde{z}(\tau),$$

thus

$$|Q(\tau, z(\tau), z(\tau-1))| \le 2 \int_0^1 M_2 s \big(2|z(\tau)| + |\tilde{z}(\tau)|\big) \, ds \cdot |z(\tau)| \\ + \int_0^1 M_2 s \big(2|z(\tau)| + |\tilde{z}(\tau)|\big) \, ds \cdot |\tilde{z}(\tau)| \\ = M_2 \big(2|z(\tau)| + |\tilde{z}(\tau)|\big)^2 / 2$$

and

$$\mathcal{I}_1 z(t) \;=\; \int_{-\infty}^{\infty} \Phi(t) G(t - \tau) \Phi^{-1}(\tau) (E - D(\tau))^{-1} Q(\tau, z(\tau), z(\tau-1)) \, d\tau \;=\; O(\mu^2). \tag{17}$$

Now let us estimate $|\dot{x}(t)|$, where $x(t)$ is a solution of (2). We have

$$|\dot{x}(t)| \leq |D(t)||\dot{x}(t)| + |f(t, x(t), x(t-h), y^h(t))| \leq \eta \sup |\dot{x}(t)| + M_0.$$

Thus

$$\sup |\dot{x}(t)| \leq \eta \sup |\dot{x}(t)| + M_0$$

and, finally,

$$\sup |\dot{x}(t)| \leq M_0(1-\eta)^{-1}.$$

Further on, since the intervals $(t_i - h, t_i)$ contain none of the points t_j, we have

$$\begin{aligned} \delta I_i(x_i, \bar{x}_i) &= \int_0^1 \frac{\partial}{\partial s} I_i(x_i, x(t_i - sh))\, ds \\ &= \int_0^1 \partial_{\bar{x}} I_i(x_i, x(t_i - sh)) \frac{\partial}{\partial s} x(t_i - sh)\, ds \\ &= -h \int_0^1 \partial_{\bar{x}}\, I_i(x_i, x(t_i - sh))\dot{x}(t_i - sh)\, ds, \end{aligned} \tag{18}$$

thus $|\delta I_i(x_i, \bar{x}_i)| \leq h M_1 M_0 (1-\eta)^{-1}$ and

$$\mathcal{S}_2 z(t) \equiv \sum_{i \in \mathbb{Z}} \Phi(t) G(t - t_i) \Phi^{-1}(t_i) \delta I_i(x_i, \bar{x}_i) = O(h). \tag{19}$$

Then we estimate the difference $x(t) - x(t-h)$. If t is not in Δ_1^h, then $x(t)$ is continuous on $[t-h, t]$ and $\dot{x}$ exists on this interval, with the possible exception of finitely many points. Then we have

$$|x(t) - x(t-h)| \leq h M_0 (1-\eta)^{-1}.$$

Now we shall obtain an analogous estimate for Δ_1^h. Then the interval $(t-h, t)$ contains just one point of discontinuity t_i of $x(t)$, thus

$$|x(t) - x(t-h)| \leq |x(t) - x(t_i + 0)| + |x(t_i + 0) - x(t_i)| + |x(t_i) - x(t-h)|$$

$$\leq M_0(1-\eta)^{-1}(t - t_i) + M_0 + M_0(1-\eta)^{-1}(t_i - t + h) = M_0(1 + h(1-\eta)^{-1}).$$

Next we estimate the difference $x(t-1) - x(t-1-h\varphi(t))$. If $t \notin \Delta_2^h$, then $x(t)$ is continuous in the interval $]t_i(h), t_i + 1[$ and $|x(t-1) - x(t-1-h\varphi(t))| \leq M_0 h (1-\eta)^{-1}$.

Let $t \in \Delta_2^h$, *i.e.*, $t \in]t_i(h), t_i + 1[$ for some $i \in \mathbb{Z}$. This means that the interval $]t-1, t-1-h\varphi(t)[$ contains just one discontinuity point t_i. Then we have

$$\begin{aligned} &|x(t-1) - x(t-1-h\varphi(t))| \leq |x(t-1) - x(t_i + \operatorname{sgn}\varphi(t))| \\ &+|x(t_i + \operatorname{sgn}\varphi(t)) - x(t_i - \operatorname{sgn}\varphi(t))| + |x(t_i - \operatorname{sgn}\varphi(t)) - x(t-1-h\varphi(t))| \\ \leq &\sup |\dot{x}(t)|(t-1-t_i)\operatorname{sgn}\varphi(t) + |I_i(x_i, \bar{x}_i)| + \sup|\dot{x}(t)|(t_i - t + 1 + h\varphi(t))\operatorname{sgn}\varphi(t) \\ = &\, h \sup |\dot{x}(t)||\varphi(t)| + |I_i(x_i, \bar{x}_i)| \leq M_0(1 + h(1-\eta)^{-1}). \end{aligned}$$

Using these estimates, we evaluate $\delta f(\tau, x(\tau), \bar{x}(\tau), y^h(\tau))$. If $\tau \in \Delta_3^h$, we have

$$\delta f(\tau, x(\tau), \bar{x}(\tau), y^h(\tau)) \;=\; \int_0^1 \frac{\partial}{\partial s} f(\tau, x(\tau), x(\tau - sh), y^{sh}(\tau))\, ds$$

$$= -h \int_0^1 \partial_{\bar{x}} f(\tau, x(\tau), x(\tau - sh), y^{sh}(\tau)) \dot{x}(\tau - sh)\, ds$$

$$-h\varphi(\tau) \int_0^1 \partial_y f(\tau, x(\tau), x(\tau - sh), y^{sh}(\tau)) \dot{x}(\tau - 1 - sh\varphi(\tau))\, ds$$

and

$$|\delta f(\tau, x(\tau), \bar{x}(\tau), y^h(\tau))| \;\leq\; 2hM_0M_1(1-\eta)^{-1}. \tag{20}$$

Next, if $\tau \in \Delta_1^h$, we have

$$\begin{aligned}
&\delta f(\tau, x(\tau), \bar{x}(\tau), y^h(\tau)) \\
&= \int_0^1 \frac{\partial}{\partial s} f(\tau, x(\tau), sx(\tau - h) + (1-s)x(\tau), y^{sh}(\tau))\, ds \\
&= \int_0^1 \partial_{\bar{x}} f(\tau, x(\tau), sx(\tau - h) + (1-s)x(\tau), y^{sh}(\tau))\, ds \cdot (x(\tau - h) - x(\tau)) \\
&\quad -h\varphi(\tau) \int_0^1 \partial_y f(\tau, x(\tau), sx(\tau - h) + (1-s)x(\tau), y^{sh}(\tau)) \dot{x}(\tau - 1 - sh\varphi(\tau))\, ds
\end{aligned}$$

and

$$\begin{aligned}
&|\delta f(\tau, x(\tau), \bar{x}(\tau), y^h(\tau))| \\
&\leq M_1\big\{M_0(1 + h(1-\eta)^{-1}) + hM_0(1-\eta)^{-1}\big\} \\
&= M_0M_1(1 + 2h(1-\eta)^{-1}).
\end{aligned} \tag{21}$$

Finally, for $\tau \in \Delta_2^h$

$$\begin{aligned}
&\delta f(\tau, x(\tau), \bar{x}(\tau), y^h(\tau)) \\
&= \int_0^1 \frac{\partial}{\partial s} f(\tau, x(\tau), x(\tau - sh), sy^h(\tau) + (1-s)y^0(\tau))\, ds \\
&= -h \int_0^1 \partial_{\bar{x}} f(\tau, x(\tau), x(\tau - sh), sy^h(\tau) + (1-s)y^0(\tau)) \cdot \dot{x}(\tau - sh)\, ds \\
&\quad + \int_0^1 \partial_y f(\tau, x(\tau), x(\tau - sh), sy^h(\tau) + (1-s)y^0(\tau))\, ds \cdot \big(x(\tau - 1 - h\varphi(\tau)) - x(\tau - 1)\big)
\end{aligned}$$

and

$$\begin{aligned}
&|\delta f(\tau, x(\tau), \bar{x}(\tau), y^h(\tau))| \\
&\leq M_1\big\{hM_0(1-\eta)^{-1} + M_0(1 + h(1-\eta)^{-1})\big\} \\
&= M_0M_1(1 + 2h(1-\eta)^{-1}).
\end{aligned} \tag{22}$$

Next, we use the representation

$$\mathcal{I}_2(t) = \int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\Phi^{-1}(\tau)(E-D(\tau))^{-1}\delta f(\tau, x(\tau), \bar{x}(\tau), y^h(\tau))\, d\tau$$
$$= \int_{\Delta_1^h} + \int_{\Delta_2^h} + \int_{\Delta_3^h} .$$

Making use of the estimates (20), (21) and (22), we find

$$|\mathcal{I}_2 z(t)| \le 2h\mathcal{M}M_0M_1(1-\eta)^{-2}\int_{\Delta_3^h} \|G(t-\tau)\|\, d\tau \tag{23}$$
$$+\mathcal{M}M_0M_1(1-\eta)^{-1}\big(1+2h(1-\eta)^{-1}\big)\int_{\Delta_1^h\cup\Delta_2^h} \|G(t-\tau)\|\, d\tau$$
$$= 2h\mathcal{M}M_0M_1(1-\eta)^{-2}\int_{-\infty}^{\infty} \|G(t-\tau)\|\, d\tau + \mathcal{M}M_0M_1(1-\eta)^{-1}\int_{\Delta_1^h\cup\Delta_2^h} \|G(t-\tau)\|\, d\tau.$$

Now we use the estimate (9). We need estimates for $\int_{\Delta_j^h} \|G(t-\tau)\|\, d\tau,\ j = 1, 2$.

We will estimate the integral for $j = 1$. The arguments are similar to those used for deriving the estimate (10). We have

$$\int_{\Delta_1^h} \|G(t-\tau)\|\, d\tau \le C\sum_{i\in\mathbb{Z}}\int_{t_i}^{t_i+h} e^{-\alpha|t-\tau|}\, d\tau.$$

For $t \in \mathbb{R}$, we shall consider two possibilities:

a) t belongs to none of the segments $[t_i, t_i+h],\ i \in \mathbb{Z}$. Then we may assume that $t_0 + h < t < t_1$. Now for $i \in \mathbb{N}$ we have

$$\int_{t_i}^{t_i+h} e^{-\alpha|t-\tau|}\, d\tau \le he^{-\alpha(t_i-t_1)} \le he^{-\alpha\theta(i-1)}.$$

For $i \notin \mathbb{N}$

$$\int_{t_i}^{t_i+h} e^{-\alpha|t-\tau|}\, d\tau \le he^{-\alpha(t_0-t_i)} \le he^{\alpha\theta i},$$

and as above we conclude that

$$\int_{\Delta_1^h} \|G(t-\tau)\|\, d\tau \le \frac{2Ch}{1-e^{-\alpha\theta}}. \tag{24}$$

b) t belongs to one of these segments, say, $t_0 \le t \le t_0 + h$. Now for $i \in \mathbb{N}$ we have

$$\int_{t_i}^{t_i+h} e^{-\alpha|t-\tau|}\, d\tau \le he^{-\alpha(t_i-t_0-h)} \le he^{\alpha h}e^{-\alpha\theta i},$$

and for $-i \in \mathbb{N}$

$$\int_{t_i}^{t_i+h} e^{-\alpha|t-\tau|}\, d\tau \le he^{-\alpha(t_0-t_i-h)} \le he^{\alpha h}e^{\alpha\theta i}.$$

Finally,

$$\int_{t_0}^{t_0+h} e^{-\alpha|t-\tau|}\,d\tau \le h$$

and

$$\int_{\Delta_1^h} \|G(t-\tau)\|\,d\tau \le Ch\left(1+2e^{\alpha h}\sum_{i=1}^{\infty} e^{-i\alpha\theta}\right) = Ch\left(1+\frac{2e^{\alpha(h-\theta)}}{1-e^{-\alpha\theta}}\right). \tag{25}$$

Combining the estimates (24) and (25), we have

$$\int_{\Delta_1^h} \|G(t-\tau)\|\,d\tau \le \frac{Ch}{1-e^{-\alpha\theta}}\max\left\{2,\ 1-e^{-\alpha\theta}+2e^{\alpha(h-\theta)}\right\}.$$

For h small enough, namely, for

$$h \le h_1 = \frac{\ln(1+e^{\alpha\theta})-\ln 2}{\alpha},$$

estimate (24) holds for any $t\in\mathbb{R}$.

In a similar way we derive the estimate

$$\int_{\Delta_2^h} \|G(t-\tau)\|\,d\tau \le \frac{2Ch}{1-e^{-\alpha\theta}} \tag{26}$$

for any $t\in\mathbb{R}$ and $h\le h_1/2$. We omit the calculations, which can be found in [11].

Substituting the estimates (9), (24) and (26) into (23), we find

$$|\mathcal{I}_2 z(t)| = O(h). \tag{27}$$

Adding together the estimates (16), (17), (19) and (27), we obtain

$$|\mathcal{U}_h z(t)| \;=\; O(\mu^2)+O(h). \tag{28}$$

Henceforth, we shall repeatedly use the following lemma or arguments of its proof.

Lemma 3.1. Let $y(t)$ be such that in each of the intervals (t_i,t_{i+1}), $i\in\mathbb{Z}$, $\dot y(t)$ exists, except for a finite number of points, and is bounded for $t\in\mathbb{R}$. Then for any $h>0$ and for any function $\chi(t)$ such that both the integral $I=\int\limits_{-\infty}^{\infty}|\chi(t)|\,dt$ and the sum $S(h)=\sum\limits_{i\in\mathbb{Z}}\sup\limits_{t\in(t_i,t_i+h)}|\chi(t)|$ are convergent we have

$$\left|\int_{-\infty}^{\infty}\chi(t)\big(y(t)-y(t-h)\big)\,dt\right| \le h\left(I\sup_{\tau\in\mathbb{R}}|\dot y(\tau)|+S(h)\sup_{i\in\mathbb{Z}}|\Delta y(t_i)|\right).$$

Proof. It suffices to prove the assertion for h small enough. Then we can define the "bad" set $\Delta_1^h=\bigcup\limits_{i\in\mathbb{Z}}(t_i,t_i+h)$ as above. If $t\notin\Delta_1^h$, then

$$|y(t)-y(t-h)| \le h\sup_{\tau\in\mathbb{R}}|\dot y(\tau)|.$$

If $t \in \Delta_1^h$, then we have

$$|y(t) - y(t-h)| \le h \sup_{\tau\in\mathbb{R}} |\dot{y}(\tau)| + \sup_{i\in\mathbb{Z}} |\Delta y(t_i)|.$$

Now we obtain

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} \chi(t)\big(y(t) - y(t-h)\big)\,dt\right| \\
&\le \int_{\mathbb{R}\setminus\Delta_1^h} |\chi(t)| h \sup_{\tau\in\mathbb{R}} |\dot{y}(\tau)|\,dt \\
&\quad + \int_{\Delta_1^h} |\chi(t)| \left(h \sup_{\tau\in\mathbb{R}} |\dot{y}(\tau)| + \sup_{i\in\mathbb{Z}} |\Delta y(t_i)|\right) dt \\
&\le h \left(\sup_{\tau\in\mathbb{R}} |\dot{y}(\tau)| \int_{-\infty}^{\infty} |\chi(t)|\,dt + \sup_{i\in\mathbb{Z}} |\Delta y(t_i)| \sum_{i\in\mathbb{Z}} \sup_{t\in(t_i,t_i+h)} |\chi(t)|\right).
\end{aligned}$$

Note that in the proof of this lemma we have used arguments which are just simplified versions of those used in the evaluation of $\mathcal{I}_2 z(t)$.

In order to estimate $-\mathcal{V}_h z(t)$, we represent the integral in (15) as a difference of two integrals and change the integration variable in the first one to obtain

$$\begin{aligned}
-\mathcal{V}_h z(t) = \int_{-\infty}^{\infty} \Phi(t)\big\{&G(t-\tau)\Phi^{-1}(\tau)(E - D(\tau))^{-1}D(\tau) \\
&-G(t-\tau+h)\Phi^{-1}(\tau-h)(E - D(\tau-h))^{-1}D(\tau-h)\big\}\bar{x}(\tau)\,d\tau.
\end{aligned} \tag{29}$$

In order to apply Lemma 3.1, we carry out one more transformation to obtain

$$\begin{aligned}
-\mathcal{V}_h z(t) = &\int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\big\{\Phi^{-1}(\tau)(E - D(\tau))^{-1}D(\tau) \\
&\qquad -\Phi^{-1}(\tau-h)(E - D(\tau-h))^{-1}D(\tau-h)\big\}\bar{x}(\tau)\,d\tau \\
&+\int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\big\{\Phi^{-1}(\tau-h)(E - D(\tau-h))^{-1}D(\tau-h)\dot{x}(\tau-h) \\
&\qquad -\Phi^{-1}(\tau)(E - D(\tau))^{-1}D(\tau)\dot{x}(\tau)\big\}\,d\tau.
\end{aligned}$$

We apply Lemma 3.1 to the first integral, with $\Phi(t)G(t-\tau)\bar{x}(\tau)$ considered as a function of τ for any fixed t instead of χ, and $\Phi^{-1}(\tau)(E - D(\tau))^{-1}D(\tau)$ instead of y (with points of discontinuity t_i, $i \in \mathbb{Z}$). Similarly, we can apply the lemma to the second integral with $\Phi(t)G(t-\tau)$ considered as a function of τ for any fixed t instead of χ, and $\Phi^{-1}(\tau)(E - D(\tau))^{-1}D(\tau)\dot{x}(\tau)$ instead of y. Thus both integrals are estimated by $O(h)$,

$$-\mathcal{V}_h z(t) = O(h)$$

and

$$|\mathcal{U}_h z(t) - \mathcal{V}_h z(t)| = O(\mu^2) + O(h),$$

i.e.,

$$|\mathcal{U}_h z(t) - \mathcal{V}_h z(t)| \le K_1\mu^2 + K_2 h \tag{30}$$

for some positive constants K_1 and K_2.

To provide the validity of the inequality $|\mathcal{U}_h z(t) - \mathcal{V}_h z(t)| \le \mu$, we first choose

$$\tilde{\mu}_0 \;=\; \min\left\{\mu_0,\; \frac{1}{2K_1}\right\}.$$

Then, for any $\mu \in (0, \tilde{\mu}_0]$, we have $K_1\mu^2 \;\le\; \mu/2$ and inequality (30) takes on the form

$$|\mathcal{U}_h z(t) - \mathcal{V}_h z(t)| \le \mu/2 \;+\; K_2 h.$$

If we choose

$$\tilde{h}(\mu) \;=\; \min\left\{h_0,\; \frac{h_1}{2},\; \frac{\mu}{2K_2}\right\},$$

then, for any $h \in (0, \tilde{h}(\mu)]$, we have $K_2 h \;\le\; \mu/2$ and thus

$$|\mathcal{U}_h z(t) - \mathcal{V}_h z(t)| \;\le\; \mu,$$

i.e., the operator $\mathcal{U}_h - \mathcal{V}_h$ maps the set $\mathcal{T}_\mu$ into itself for $\mu \in (0, \tilde{\mu}_0]$ and $h \in (0, \tilde{h}(\mu)]$. □

3.3 Contraction property of the operator $\mathcal{U}_h - \mathcal{V}_h$

Let $z', z'' \in \mathcal{T}_\mu$. Then

$$\begin{aligned}\mathcal{U}_h z'(t) - \mathcal{U}_h z''(t) = {} & (\mathcal{I}_1 z'(t) - \mathcal{I}_1 z''(t)) \;+\; (\mathcal{I}_2 z'(t) - \mathcal{I}_2 z''(t)) \\ & + (\mathcal{S}_1 z'(t) \mathcal{S}_1 z''(t)) \;+\; (\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t)).\end{aligned}$$

First we consider

$$\mathcal{S}_1 z'(t) - \mathcal{S}_1 z''(t) \;=\; \sum_{i\in\mathbb{Z}} \Phi(t) G(t-t_i) \Phi^{-1}(t_i)\big(J_i(z'_i) \;-\; J_i(z''_i)\big).$$

We have

$$\begin{aligned} & J_i(z'_i) \;-\; J_i(z''_i) \\ = {} & \big(I_i(\psi_i + z'_i, \psi_i + z'_i) \;-\; I_i(\psi_i + z''_i, \psi_i + z''_i)\big) \;-\; B_i(z'_i - z''_i) \\ = {} & \left\{\int_0^1 \big(\partial_x I_i(\psi_i + s z'_i + (1-s) z''_i, \psi_i + s z'_i + (1-s) z''_i) \;-\; \partial_x I_i(\psi_i, \psi_i)\big)\, ds \right. \\ & \left. +7 \int_0^1 \big(\partial_{\bar{x}} I_i(\psi_i + s z'_i + (1-s) z''_i, \psi_i + s z'_i + (1-s) z''_i) \;-\; \partial_{\bar{x}} I_i(\psi_i, \psi_i)\big)\, ds \right\} (z'_i - z''_i), \end{aligned}$$

thus

$$\begin{aligned}|J_i(z'_i) \;-\; J_i(z''_i)| \le {} & \; 4M_2 \int_0^1 \big[s|z'_i| + (1-s)|z''_i|\big]\, ds \cdot |z'_i - z''_i| \\ \le {} & \; 2M_2\big(|z'_i| + |z''_i|\big)|z'_i - z''_i| \;\le\; 4\mu M_2 |z'_i - z''_i|\end{aligned}$$

and

$$|\mathcal{S}_1 z'(t) \; - \; \mathcal{S}_1 z''(t)| \;\le\; O(\mu)\|z' \; - \; z''\|. \tag{31}$$

Next,

$$\mathcal{I}_1 z'(t) - \mathcal{I}_1 z''(t) = \int_{-\infty}^{\infty} \Phi(t)G(t-\tau)\Phi^{-1}(\tau)(E - D(\tau))^{-1} \times \\ \times\big(Q(\tau, z'(\tau), \tilde{z}'(\tau)) - Q(\tau, z''(\tau), \tilde{z}''(\tau))\big)d\tau.$$

We have

$$\begin{aligned} &Q(\tau, z'(\tau), \tilde{z}'(\tau)) - Q(\tau, z''(\tau), \tilde{z}''(\tau)) \\ &= \Big\{\int_0^1 \Big[\partial_x f(\tau, x_s(\tau), x_s(\tau), \tilde{x}_s(\tau)) - \partial_x f(\tau, \psi(\tau), \psi(\tau), \tilde{\psi}(\tau))\Big]\, ds \\ &\quad + \int_0^1 \Big[\partial_{\bar{x}} f(\tau, x_s(\tau), x_s(\tau), \tilde{x}_s(\tau)) - \partial_{\bar{x}} f(\tau, \psi(\tau), \psi(\tau), \tilde{\psi}(\tau))\Big]\, ds\Big\} \cdot \big(z'(\tau) - z''(\tau)\big) \\ &\quad + \int_0^1 \Big[\partial_y f(\tau, x_s(\tau), x_s(\tau), \tilde{x}_s(\tau)) - \partial_y f(\tau, \psi(\tau), \psi(\tau), \tilde{\psi}(\tau))\Big]\, ds \cdot \big(\tilde{z}''(\tau) - \tilde{z}''(\tau)\big), \end{aligned}$$

where $x_s(\tau) = \psi(\tau) + sz'(\tau) + (1-s)z''(\tau)$. Thus

$$\begin{aligned} &|Q(\tau, z'(\tau), \tilde{z}'(\tau)) - Q(\tau, z''(\tau), \tilde{z}''(\tau))| \\ &\le 2M_2 \int_0^1 \big[s\big(2|z'(\tau)| + |\tilde{z}'(\tau)|\big) \;+\; (1-s)\big(2|z''(\tau)| + |\tilde{z}''(\tau)|\big)\big]\, ds \cdot |z'(\tau) - z''(\tau)| \\ &\quad + M_2 \int_0^1 \big[s\big(2|z'(\tau)| + |\tilde{z}'(\tau)|\big) \;+\; (1-s)\big(2|z''(\tau)| + |\tilde{z}''(\tau)|\big)\big]\, ds \cdot |\tilde{z}'(\tau) - \tilde{z}''(\tau)| \\ &\le 2M_2 \frac{1}{2} \cdot 3\big(\|z'\| + \|z''\|\big) \cdot \|z' - z''\| + M_2 \frac{1}{2} \cdot 3\big(\|z'\| + \|z''\|\big) \cdot \|z' - z''\| \\ &\le \; 9\mu M_2 \, \|z' \; - \; z''\| \end{aligned}$$

and

$$|\mathcal{I}_1 z'(t) \; - \; \mathcal{I}_1 z''(t)| \;\le\; O(\mu)\|z' \; - \; z''\|. \tag{32}$$

For the estimation of $\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t)$ and $\mathcal{I}_2 z'(t) - \mathcal{I}_2 z''(t)$ we denote $x' = \psi(t) + z'$, $x'' = \psi(t) + z''$, $y'^h(t) = x'(t - 1 - h\varphi(t))$, etc. Now

$$\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t) \;=\; \sum_{i\in\mathbb{Z}} \Phi(t)G(t - t_i)\Phi^{-1}(t_i)\big(\delta I_i(x_i', \bar{x}_i') \;-\; \delta I_i(x_i'', \bar{x}_i'')\big)$$

and

$$\begin{aligned}\delta I_i(x_i',\bar x_i') - \delta I_i(x_i'',\bar x_i'') &= \big(I_i(x_i',\bar x_i') - I_i(x_i',x_i')\big) - \big(I_i(x_i'',\bar x_i'') - I_i(x_i'',x_i'')\big)\\ &= \big(I_i(x_i',\bar x_i') - I_i(\psi_i,\psi_i)\big) - \big(I_i(x_i',x_i') - I_i(\psi_i,\psi_i)\big)\\ &\quad -\big(I_i(x_i'',\bar x_i'') - I_i(\psi_i,\psi_i)\big) + \big(I_i(x_i'',x_i'') - I_i(\psi_i,\psi_i)\big)\\ &= -\int_0^1 \frac{\partial}{\partial s} I_i(\psi_i + sz_i', \psi_i + s(\bar\psi_i - \psi_i + \bar z_i'))\, d(1-s)\\ &\quad + \int_0^1 \frac{\partial}{\partial s} I_i(\psi_i + sz_i', \psi_i + sz_i')\, d(1-s)\\ &\quad + \int_0^1 \frac{\partial}{\partial s} I_i(\psi_i + sz_i'', \psi_i + s(\bar\psi_i - \psi_i + \bar z_i''))\, d(1-s)\\ &\quad - \int_0^1 \frac{\partial}{\partial s} I_i(\psi_i + sz_i'', \psi_i + sz_i'')\, d(1-s).\end{aligned}$$

Making use (for the first time in this paper) of the continuity of the second derivatives of $I_i(x,\bar x)$ (condition A3), we integrate by parts and rearrange the terms to obtain

$$\begin{aligned}&\delta I_i(x_i',\bar x_i') - \delta I_i(x_i'',\bar x_i'')\\ &= \Big\{\Big\langle \int_0^1 (1-s)\frac{\partial^2}{\partial x^2} I_i(x,x)|_{x=\psi_i+sz_i''}\, ds\cdot z_i'', z_i''\Big\rangle\\ &\quad - \Big\langle \int_0^1 (1-s)\frac{\partial^2}{\partial x^2} I_i(x,x)|_{x=\psi_i+sz_i'}\, ds\cdot z_i', z_i'\Big\rangle\Big\}\\ &\quad + \Big\{\Big\langle \int_0^1 (1-s)\partial^2_{xx} I_i(\psi_i+sz_i', \psi_i + s(\bar\psi_i-\psi_i+\bar z_i'))\, ds\cdot z_i', z_i'\Big\rangle\\ &\quad - \Big\langle \int_0^1 (1-s)\partial^2_{xx} I_i(\psi_i+sz_i'', \psi_i + s(\bar\psi_i-\psi_i+\bar z_i''))\, ds\cdot z_i'', z_i''\Big\rangle\Big\}\\ &\quad +2\Big\{\Big\langle \int_0^1 (1-s)\partial^2_{x\bar x} I_i(\psi_i+sz_i', \psi_i + s(\bar\psi_i-\psi_i+\bar z_i'))\, ds\cdot z_i', \bar\psi_i-\psi_i+\bar z_i'\Big\rangle\\ &\quad - \Big\langle \int_0^1 (1-s)\partial^2_{x\bar x} I_i(\psi_i+sz_i'', \psi_i + s(\bar\psi_i-\psi_i+\bar z_i''))\, ds\cdot z_i'', \bar\psi_i-\psi_i+\bar z_i''\Big\rangle\Big\}\\ &\quad + \Big\{\Big\langle \int_0^1 (1-s)\partial^2_{\bar x\bar x} I_i(\psi_i+sz_i', \psi_i + s(\bar\psi_i-\psi_i+\bar z_i'))\, ds\cdot (\bar\psi_i-\psi_i+\bar z_i'), \bar\psi_i-\psi_i+\bar z_i'\Big\rangle\\ &\quad - \Big\langle \int_0^1 (1-s)\partial^2_{\bar x\bar x} I_i(\psi_i+sz_i'', \psi_i + s(\bar\psi_i-\psi_i+\bar z_i''))\, ds\cdot (\bar\psi_i-\psi_i+\bar z_i''), \bar\psi_i-\psi_i+\bar z_i''\Big\rangle\Big\}.\end{aligned}$$

Now we estimate separately the four addends in the braces making use also of the Lipschitz continuity of the second derivatives of $I_i(x_i,\bar x_i)$ according to condition A3. It is easy to see that the first two addends are estimated by $O(\mu)\|z'-z''\|$, while the other two terms are estimated by $(O(\mu)+O(h))\|z'-z''\|$. So we obtain

$$|\mathcal{S}_2 z'(t) - \mathcal{S}_2 z''(t)| \le (O(\mu)+O(h))\|z' - z''\|. \tag{33}$$

We estimate $\mathcal{I}_2 z'(t) - \mathcal{I}_2 z''(t)$ in a similar way, using condition A1. Now we have (the argument τ is dropped for brevity)

$$
\begin{aligned}
&\delta f(\cdot, x', \bar{x}', y'^h) - \delta f(\cdot, x'', \bar{x}'', y''^h) \\
= &\left\{ \left\langle \int_0^1 (1-s) \frac{\partial^2}{\partial x^2} f(\cdot, x, x, \tilde{\psi} + s\tilde{z}'')|_{x=\psi+sz''}\, ds \cdot z'', z'' \right\rangle \right. \\
&\left. - \left\langle \int_0^1 (1-s) \frac{\partial^2}{\partial x^2} f(\cdot, x, x, \tilde{\psi} + s\tilde{z}')|_{x=\psi+sz'}\, ds \cdot z', z' \right\rangle \right\} \\
&+2 \left\{ \left\langle \int_0^1 (1-s) \frac{\partial^2}{\partial x \partial y} f(\cdot, x, x, y) \Bigg|_{\substack{x=\psi+sz'' \\ y=\tilde{\psi}+s\tilde{z}''}} ds \cdot z'', \tilde{z}'' \right\rangle \right. \\
&\left. - \left\langle \int_0^1 (1-s) \frac{\partial^2}{\partial x \partial y} f(\cdot, x, x, y) \Bigg|_{\substack{x=\psi+sz' \\ y=\tilde{\psi}+s\tilde{z}'}} ds \cdot z', \tilde{z}' \right\rangle \right\} \\
&+ \left\{ \left\langle \int_0^1 (1-s) \partial^2_{\bar{x}\bar{x}} f(\cdot, \psi + sz'', \psi + sz'', \tilde{\psi} + s\tilde{z}'')\, ds \cdot z'', z'' \right\rangle \right. \\
&\left. - \left\langle \int_0^1 (1-s) \partial^2_{\bar{x}\bar{x}} f(\cdot, \psi + sz', \psi + sz', \tilde{\psi} + s\tilde{z}')\, ds \cdot z', z' \right\rangle \right\} \\
&+ \left\{ \left\langle \int_0^1 (1-s) \partial^2_{xx} f(\cdot, \psi+sz', \psi+s(\bar{\psi}-\psi+\bar{z}'), \tilde{\psi}+s(\psi^h-\psi^0+z'^h))\, ds \cdot z', z' \right\rangle \right. \\
&\left. - \left\langle \int_0^1 (1-s) \partial^2_{xx} f(\cdot, \psi+sz'', \psi+s(\bar{\psi}-\psi+\bar{z}''), \tilde{\psi}+s(\psi^h-\psi^0+z''^h))\, ds \cdot z'', z'' \right\rangle \right\} \\
&+2 \left\{ \left\langle \int_0^1 (1-s) \partial^2_{x\bar{x}} f(\cdot, \psi + sz', \psi + s(\bar{\psi} - \psi + \bar{z}'), \tilde{\psi} + s(\psi^h - \psi^0 + z'^h))\, ds \cdot z', \right.\right. \\
&\qquad\qquad \left. \bar{\psi} - \psi + \bar{z}' \right\rangle \\
&- \left\langle \int_0^1 (1-s) \partial^2_{x\bar{x}} f(\cdot, \psi + sz'', \psi + s(\bar{\psi} - \psi + \bar{z}''), \tilde{\psi} + s(\psi^h - \psi^0 + z''^h))\, ds \cdot z'', \right. \\
&\qquad\qquad \left.\left. \bar{\psi} - \psi + \bar{z}'' \right\rangle \right\} \\
&+ \left\{ \left\langle \int_0^1 (1-s) \partial^2_{\bar{x}\bar{x}} f(\cdot, \psi + sz', \psi + s(\bar{\psi} - \psi + \bar{z}'), \tilde{\psi} + s(\psi^h - \psi^0 + z'^h))\, ds \cdot \right.\right. \\
&\qquad\qquad \left. (\bar{\psi} - \psi + \bar{z}'), \bar{\psi} - \psi + \bar{z}' \right\rangle \\
&- \left\langle \int_0^1 (1-s) \partial^2_{\bar{x}\bar{x}} f(\cdot, \psi + sz'', \psi + s(\bar{\psi} - \psi + \bar{z}''), \tilde{\psi} + s(\psi^h - \psi^0 + z''^h))\, ds \cdot \right. \\
&\qquad\qquad \left.\left. (\bar{\psi} - \psi + \bar{z}''), \bar{\psi} - \psi + \bar{z}'' \right\rangle \right\}
\end{aligned}
$$

$$+2\left\{\left\langle\int_0^1(1-s)\partial^2_{xy}f(\cdot,\psi+sz',\psi+s(\bar\psi-\psi+\bar z'),\tilde\psi+s(\psi^h-\psi^0+z'^h))\,ds\cdot z',\right.\right.$$
$$\left.\psi^h-\psi^0+z'^h\right\rangle$$

$$-\left\langle\int_0^1(1-s)\partial^2_{xy}f(\cdot,\psi+sz'',\psi+s(\bar\psi-\psi+\bar z''),\tilde\psi+s(\psi^h-\psi^0+z''^h))\,ds\cdot z'',\right.$$
$$\left.\left.\psi^h-\psi^0+z''^h\right\rangle\right\}$$

$$+2\left\{\left\langle\int_0^1(1-s)\partial^2_{\bar xy}f(\cdot,\psi+sz',\psi+s(\bar\psi-\psi+\bar z'),\tilde\psi+s(\psi^h-\psi^0+z'^h))\,ds\cdot\right.\right.$$
$$\left.(\bar\psi-\psi+\bar z'),\psi^h-\psi^0+z'^h\right\rangle$$

$$-\left\langle\int_0^1(1-s)\partial^2_{\bar xy}f(\cdot,\psi+sz'',\psi+s(\bar\psi-\psi+\bar z''),\tilde\psi+s(\psi^h-\psi^0+z''^h))\,ds\cdot\right.$$
$$\left.\left.(\bar\psi-\psi+\bar z''),\psi^h-\psi^0+z''^h\right\rangle\right\}$$

$$+\left\{\left\langle\int_0^1(1-s)\partial^2_{yy}f(\cdot,\psi+sz',\psi+s(\bar\psi-\psi+\bar z'),\tilde\psi+s(\psi^h-\psi^0+z'^h))\,ds\cdot\right.\right.$$
$$\left.(\psi^h-\psi^0+z'^h),\psi^h-\psi^0+z'^h\right\rangle$$

$$-\left\langle\int_0^1(1-s)\partial^2_{yy}f(\cdot,\psi+sz'',\psi+s(\bar\psi-\psi+\bar z''),\tilde\psi+s(\psi^h-\psi^0+z''^h))\,ds\cdot\right.$$
$$\left.\left.(\psi^h-\psi^0+z''^h),\psi^h-\psi^0+z''^h\right\rangle\right\}.$$

The first four expressions in the braces are estimated by $O(\mu)\|z'-z''\|$ for all $\tau\in\mathbb{R}$. The fifth and sixth expressions are estimated by

$$\begin{cases}(O(\mu)+O(h))\|z'-z''\| & \text{for } \ \tau\notin\Delta_1^h,\\ (O(\mu)+O(1))\|z'-z''\| & \text{for } \ \tau\in\Delta_1^h.\end{cases}$$

The seventh and ninth expressions are estimated by

$$\begin{cases}(O(\mu)+O(h))\|z'-z''\| & \text{for } \ \tau\notin\Delta_2^h,\\ (O(\mu)+O(1))\|z'-z''\| & \text{for } \ \tau\in\Delta_2^h.\end{cases}$$

Finally, the eighth expression is estimated by

$$\begin{cases}(O(\mu)+O(h))\|z'-z''\| & \text{for } \ \tau\in\Delta_3^h,\\ (O(\mu)+O(1))\|z'-z''\| & \text{for } \ \tau\in(\Delta_1^h\cup\Delta_2^h).\end{cases}$$

Using these estimates, by arguments similar to those in the proof of Lemma 3.1 we find

$$|\mathcal{I}_2z'(t)-\mathcal{I}_2z''(t)|\le(O(\mu)+O(h))\|z'-z''\|. \tag{34}$$

Now by virtue of the estimates (31), (32), (33) and (34) we obtain

$$\|\mathcal{U}_hz'-\mathcal{U}_hz''\|\le(O(\mu)+O(h))\|z'-z''\|.$$

In order to estimate $\mathcal{V}_h z' - \mathcal{V}_h z''$, we integrate by parts the expression (29) for $\mathcal{V}_h z(t)$, taking into account that the function $G(t-\tau)$ is discontinuous at $\tau = t$ while $\Phi(\tau)$ and $x(\tau)$ are discontinuous at $t_1, \dots, t_m$ and making use of the equalities

$$\frac{\partial}{\partial \tau}\big[\Phi(t)G(t-\tau)\Phi^{-1}(\tau)\big] = -\Phi(t)G(t-\tau)\Phi^{-1}(\tau)(E-D(\tau))^{-1}A(\tau)$$

and $\Phi(t_i+0) = (E+B_i)\Phi(t_i),\ G(+0) - G(-0) = E$. We obtain

$$\begin{aligned}
&\mathcal{V}_h z(t) \\
&= (E-D(t))^{-1}D(t)\big(x(t) - x(t-h)\big) \\
&\quad + \sum_{i\in\mathbf{Z}} \Phi(t)\Big\{G(t-t_i-h)\Phi^{-1}(t_i+h)(E-D(t_i+h))^{-1}D(t_i+h) \\
&\quad - G(t-t_i)\Phi^{-1}(t_i+0)(E-D(t_i))^{-1}D(t_i)\Big\} I_i(x_i,\bar{x}_i) \\
&\quad + \sum_{i\in\mathbf{Z}} \Phi(t)G(t-t_i)\Phi^{-1}(t_i+0)B_i(E-D(t_i))^{-1}D(t_i)(x_i-\bar{x}_i) \\
&\quad - \int_{-\infty}^{\infty} \Phi(t)\Big\{G(t-\tau)\Phi^{-1}(\tau)(E-D(\tau))^{-1}A(\tau)(E-D(\tau))^{-1}D(\tau) \\
&\quad - G(t-\tau+h)\Phi^{-1}(\tau-h)(E-D(\tau-h))^{-1}A(\tau-h)(E-D(\tau-h))^{-1}D(\tau-h)\Big\}x(\tau-h)\,d\tau \\
&\quad + \int_{-\infty}^{\infty} \Phi(t)\Big\{G(t-\tau)\Phi^{-1}(\tau)(E-D(\tau))^{-1}\dot{D}(\tau)(E-D(\tau))^{-1} \\
&\quad - G(t-\tau+h)\Phi^{-1}(\tau-h)(E-D(\tau-h))^{-1}\dot{D}(\tau-h)(E-D(\tau-h))^{-1}\Big\}x(\tau-h)\,d\tau.
\end{aligned}$$

Further transforming the two integral terms and applying Lemma 3.1 or arguments of its proof, we see that they are estimated by $O(h)\|x\|$. The sum of the first and third terms can be estimated by $2(1+\mathcal{M}\beta)\eta(1-\eta)^{-1}\|x\|$.

The difference

$$G(t-t_i-h)\Phi^{-1}(t_i+h)(E-D(t_i+h))^{-1}D(t_i+h) - G(t-t_i)\Phi^{-1}(t_i+0)(E-D(t_i))^{-1}D(t_i)$$

is estimated by $O(h)$ for $t \notin \Delta_1^h$, and by $O(h) + \eta(1-\eta)^{-1}$ otherwise. At last, similarly to (33), we note that

$$|I_i(x_i', \bar{x}_i') - I_i(x_i'', \bar{x}_i'')| \le \big(O(\mu) + O(h)\big)\|z' - z''\|,$$

thus we have

$$|\mathcal{V}_h z'(t) - \mathcal{V}_h z''(t)| \le \big(2(1+\mathcal{M}\beta)\eta(1-\eta)^{-1} + O(\mu) + O(h)\big)\|z'-z''\|$$

and

$$\begin{aligned}
&|\big(\mathcal{U}_h z'(t) - \mathcal{V}_h z'(t)\big) - \big(\mathcal{U}_h z''(t) - \mathcal{V}_h z''(t)\big)| \\
&\le \big(2(1+\mathcal{M}\beta)\eta(1-\eta)^{-1} + \gamma_1\mu + \gamma_2 h\big)\|z'-z''\|,
\end{aligned}$$

where γ_1 and γ_2 are some positive constants.

By condition (11) we have

$$\tilde{\eta} \equiv 2(1+\mathcal{M}\beta)\eta(1-\eta)^{-1} < 1.$$

Choose a number $q \in (\tilde{\eta}, 1)$ and denote $r = q - \tilde{\eta}$, and $\mu_1 = \min\{\tilde{\mu}_0, \frac{r}{2\gamma_1}\}$ and $h_1 = \min\{\tilde{h}(\mu_1), \frac{r}{2\gamma_2}\}$. Then for any $\mu \in (0, \mu_1]$ and $h \in [0, h_1]$ we have

$$\|(\mathcal{U}_h z' - \mathcal{V}_h z') - (\mathcal{U}_h z'' - \mathcal{V}_h z'')\| \le q\|z' - z''\|, \qquad q \in (0,1),$$

for any $z', z'' \in \mathcal{T}_\mu$. □

Thus the operator $\mathcal{U}_h - \mathcal{V}_h$ has a unique fixed point in $\mathcal{T}_\mu$, which is an almost periodic solution $z(t,h)$ of system (13). Since $z \equiv 0$ is the unique almost periodic solution of system (13) for $h = 0$, then $z(t,0) \equiv 0$. Now $x(t,h) = \psi(t) + z(t,h)$ is the unique almost periodic solution of system (2) and $x(t,0) = \psi(t)$. This completes the proof of Theorem 2.1. □

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