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Recent advances in the numerical modeling of constitutive relations

M.S. Gadala

Department of Mechanical Engineering, The University of British Columbia, 2324 Main Hall, Vancouver, British Columbia, Canada V6T-124

Abstract

In this paper we present an overview of the recent developments in the area of numerical and finite element modeling of nonlinear constitutive relations. The paper discusses elastic, hyperelastic, elastoplastic and anisotropic plastic material models. In the hyperelastic model an emphasis is given to the method by which the incompressibility constraint is applied. A systematic and general procedure for the numerical treatment of hyperelastic model is presented. In the elastoplastic model both infinitesimal and large strain cases are discussed. Various concerns and implications in extending infinitesimal theories into large strain case are pointed out. In the anisotropic elastoplastic case, emphasis is given to the practicality of proposed theories and its feasible and economical use in the finite element environment.

Keywords: Constitutive equations; Rubber-like materials; Hyperelasticity; Anisotropy; Elastoplastic materials; Bubble function; Reduced-selective integration; Incompressibility; Large strain; Composite materials

1. Elastic materials

A simple and commonly used constitutive law for elastic materials under large deformations is expressed as

$$S_{ij} = D_{ijkl}E_{kl} \tag{1.1}$$

and in incremental form

$$\Delta S_{ij} = D_{ijkl}\Delta E_{kl}, \tag{1.2}$$

where S_{ij} are the components of the second Piola–Kirchhoff stress tensor, E_{ij} are the components of the Green–Lagrange strain tensor and D_{ijkl} are the components of the fourth-order stress strain constitutive tensor. The components of the constitutive tensor $\boldsymbol{D}$ may be constants or dependent on the strain state. If the components are taken to be constants, i.e., identical to Hooke's law, the relation simply represents a generalization to Hooke's law. It should be noted, however, that the second Piola–Kirchhoff stress tensor and the Green–Langrange strain tensor are conjugate pairs with their scalar product giving energy per unit undeformed volume. Another characteristic of

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these tensors is being invariant under rigid body rotations. This observation makes Eqs. (1.1) and (1.2) suitable for use in large deformations if the strains are still small, i.e., only for large rotation and large displacements. The generalization of such approach to large strain problems is not strictly correct.

In large strain situation the components of the constitutive tensor will be normally function of the strain state turning the equations to be nonlinear. We consider a specific case of obtaining such components in the following section. It should be noted also that the kinematic formulation of the problem will impact on the use of Eqs. (1.1) and (1.2). Depending on the frame of reference of the kinematic formulation, the above equations may have to be transformed to the same reference frame, e.g., to an updated or current configuration if an updated Lagrangian formulation or an Eulerian formulation is used, respectively. We will discuss such transformation and appropriate stress rates in our handling of elastoplastic models.

2. Hyperelastic materials

2.1. General considerations

For hyperelastic materials a strain energy density function W exists and for an isotropic behavior this function may be considered dependent on I_1, I_2 and I_3, which are the invariants of the Cauchy–Green deformation tensor, $\boldsymbol{C}$. For an incompressible elastic behavior, however, the third strain invariant is unity and the strain energy density may be considered as function of only I_1 and I_2. For a general problem statement, we assume a body with an initial volume V, fixed along part of its boundary, A_u and has a prescribed surface traction on a part of its boundary, A_σ. The total potential of the body may be expressed as

$$\Pi = \int_V W(I_1, I_2)\mathrm{d}V - \int_V \boldsymbol{b}\cdot\boldsymbol{u}\,\mathrm{d}V - \int_{A_\sigma} \boldsymbol{T}\cdot\boldsymbol{u}\,\mathrm{d}A, \tag{2.1}$$

where $\boldsymbol{b}$ is the prescribed body forces per unit undeformed volume, $\boldsymbol{T}$ is the prescribed surface traction per unit undeformed surface area, and $\boldsymbol{u}$ is the deformation field. The resulting deformation field $\boldsymbol{u}$ is a stationary point of the strain energy density function and should satisfy the incompressibility constraint

$$|\boldsymbol{F}| - 1 = 0, \tag{2.2}$$

where $\boldsymbol{F}$ is the deformation gradient, $\boldsymbol{F} = \boldsymbol{x}\partial/\partial\boldsymbol{X}$, and $\boldsymbol{x}, \boldsymbol{X}$ are the current and initial position vectors of the particle, respectively.

For the hyperelastic material behavior, the second Piola–Kirchhoff stress tensor $\boldsymbol{S}$ and the constitutive relation tensor $\boldsymbol{D}$ are defined by

$$\boldsymbol{S} = \frac{\partial W}{\partial \boldsymbol{E}} \quad \text{or} \quad S_{ij} = \frac{\partial W}{\partial E_{ij}} \tag{2.3}$$

and

$$\boldsymbol{D} = \frac{\partial \boldsymbol{S}}{\partial \boldsymbol{E}} = \frac{\partial^2 W}{\partial \boldsymbol{E}\,\partial \boldsymbol{E}} \quad \text{or} \quad D_{ijkl} = \frac{\partial^2 W}{\partial E_{ij}\partial E_{kl}}, \tag{2.4}$$

where S_{ij} are the components of the second Piola–Kirchhoff stress tensor, E_{ij} are the components of the Green–Lagrange strain tensor and D_{ijkl} are the components of the fourth-order stress strain constitutive tensor, and

$$\Delta S_{ij} = D_{ijkl} \Delta E_{kl}, \tag{2.5}$$

The main concern in the numerical solution of hyperelastic material models is twofold; the form of the strain energy density function and the method by which the nonlinear constraint equation is going to be applied. In what follows we summarize recent achievements in the two points and highlight problem areas and concerns.

2.2. Form of the strain energy function

Many forms characterizing the strain energy function for hyperelastic and rubber-like materials exist [1–9]. The two-parameter model, commonly referred to as the Mooney–Rivlin model [8, 9] has been widely used and well established. Although for uniaxial extension and simple shear this model is not a bad approximation, it gives poor correlation in biaxial experiments. It has been found that the model is appropriate for a fairly large range of deformation of certain natural and vulcanized rubbers (undergoing strains up to 500%), but certain natural rubbers show poor correlation with the Mooney–Rivlin model.

Based on more extensive experiments with rubbers, researchers developed various other types of strain energy functions. These range from higher-order polynomial types, e.g., the Swanson model [5], the generalized Mooney–Rivlin model [6], and the modified generalized Mooney–Rivlin model [4], to exponential and logarithmic forms, e.g., the Alexander model [7] and Blatz-Ko model [1]. The use of these higher order models, generally improves the correlation between experiment and model responses for a wider range of rubber-like materials, but it entails the disadvantage of more experimental constants to describe the model. When the application and the use of the material involves thermal effects, such disadvantage become more severe and practically difficult to perform. Other forms of strain energy function for rubber-like materials involve the use of principal stretches or stretch ratios rather than the more general strain invariants [10]. Although this effort has been quite successful, its implementation in large scale finite element programs is less convenient than the traditional one of using strain invariants.

The stress and constitutive matrix calculation for various forms of strain energy model may be cast in general forms that is easily amenable in existing finite element programs. These forms are given by the following equations [2, 3]:

$$S_{ij} = \alpha_1 \delta_{ij} + \alpha_2 C_{ij}^{\wedge} + \alpha_3 H_{ij}, \tag{2.6}$$

where α_1, α_2 and α_3 are material constants depending on the choice of strain energy function, and

$$H_{ij} = \tfrac{1}{2} e_{imp} e_{jnq} C_{mn} C_{pq}, \tag{2.7}$$

$$C_{ij}^{\wedge} = I_1 \delta_{ij} - C_{ij}, \tag{2.8}$$

where e_{ijk} is the permutation tensor, C_{ij} is the Cauchy–Green deformation tensor and δ_{ij} is the Kronecker delta. The components of $\boldsymbol{H}$ can be formed by cyclic permutation of indices in the expression

$$H_{11} = C_{22}C_{33} - C_{23}C_{32}, \tag{2.9}$$

$$H_{23} = C_{31}C_{12} - C_{11}C_{32}. \tag{2.10}$$

In a similar way the constitutive relation tensor may be expressed in the following general form:

$$D_{ijkl} = \beta_1 \delta_{ij}\delta_{kl} + \beta_2 C_{ij}^{\wedge} C_{kl}^{\wedge} + \beta_3 H_{ij}H_{kl} + \beta_4 Q_{ijkl} + \beta_5 Q_{ijkl}^{\wedge} + \beta_6 C_{ijkl}^{*} + \beta_7 H_{ijkl}^{*} + \beta_8 W_{ijkl}^{\wedge}, \tag{2.11}$$

where $\beta_1, \beta_2, \ldots, \beta_8$ are material constants depending on the choice of strain energy function, and

$$Q_{ijmn} = e_{imp}e_{jnq}C_{pq}, \tag{2.12}$$

$$Q_{ijmn}^{\wedge} = \delta_{ij}\delta_{mn} - \delta_{im}\delta_{jn}, \tag{2.13}$$

$$C_{ij}^{*} = \tfrac{1}{2}[C_{ij}^{\wedge}\delta_{kl} + C_{kl}^{\wedge}\delta_{ij}], \tag{2.14}$$

$$H_{ijkl}^{*} = \tfrac{1}{2}[H_{ij}\delta_{kl} + H_{kl}\delta_{ij}], \tag{2.15}$$

$$W_{ijkl}^{\wedge} = \tfrac{1}{2}[H_{ij}C_{kl}^{\wedge} + H_{kl}C_{ij}^{\wedge}]. \tag{2.16}$$

The above equations present a unified numerical treatment for handling various forms of constitutive relations for hyperelastic and rubber-like materials. The general forms presented for calculating the stress and constitutive tensors are only dependent on the knowledge of the Cauchy–Green deformation tensor at the Gauss point. Details of the coefficients and derivation may be found in Refs. [2, 3]. Other expressions for the strain energy density function may be cast in the same unified treatment.

2.3. *Imposition of incompressibility constraint*

Another major concern in the solution of hyperelastic material problems is the method by which the nonlinear constraint equation (2.2) is applied and satisfied. It is not our intention to discuss all possible methods available in the literature. Rather, we provide some assessment for some of the commonly used methods and those that may be easily implemented or available in commercial finite element programs. Methods that will be discussed are the multi-field or dual principle, the reduced-selective integration penalty (RIP), the average constraint and the nonlinear bubble function.

2.3.1. *Multi-field or dual principle*

Many authors have used dual principles to account for incompressibility in finite element energy formulation [2, 6, 11, 12]. Much of the work on the subject stems from the early work of Reissner and the extensive treatment of Veubeke [12]. The exact use of the Hellinger–Reissner principle in finite element applications is, however, uneconomical as it introduces excessive number of unknowns. A restricted form in which the displacements and only a single stress, the pressure, is used. In such case, the mixed variational formulation will be to find a stationary point $(\boldsymbol{u}, p)$ for the functional

$$W_p = W + \int_V p(I_3 - I)\,\mathrm{d}V, \tag{2.17}$$

where W is given by Eq. (2.1). Average incompressibility is achieved if the pressure is assumed uniform over the element. If the interpolation function of the pressure is the same as that for displacement, complete incompressibility is obtained. Higher interpolation function for the pressure, than the displacement, leads to redundant unknowns and results in singular global matrix. Average incompressibility amounts to *weakening* of the constraint equation (2.2), which provides additional flexibility to this model.

Some of the advantages of using multi-field principles may be summarized as eliminating the difficulty of using $v \to 0.5$, relaxing the continuity requirement on the interpolation function of the primary variable and allowing the freedom of choosing different interpolation function for the primary and secondary variables. On the other hand, multi-field principles require additional computational effort in solving larger number of equations and the expanded global stiffness matrix is not always positive definite even if that for the original problem is. In addition, standard patch test and convergence criteria are not generally available, and the generalization to nonlinear problems is not always apparent. These disadvantages contribute strongly to the limited use of these multi-field principles in commercial finite element programs.

2.32. Reduced selective integration penalty (RIP)

In this method, the problem may be stated as finding a stationary point $\boldsymbol{u}$ for the functional

$$W_r = W + (1/2\varepsilon)\int_V (I_3 - I)^2\,\mathrm{d}V, \tag{2.18}$$

where W is given by Eq. (2.1). It is shown that for the solution of Eq. (2.18) to converge, a corresponding solution of Eqs. (2.1) and (2.2) has to be *weakened* [13, 14], e.g., by reduced integration. A study of this approach shows that there exists a condition that has to be satisfied for convergence and stability. This condition is referred to as the Ladyshenskaya–Brezzi–Babuska or the LBB condition [15, 16]. The LBB condition requires that the discrete approximation of the constraint be bounded as linear operator mapping the space of pressure approximation onto the dual of the space of velocity approximation. Unfortunately, the LBB condition is shown to be satisfied only in specific cases. These are:

- six-node quadratic triangle and nine-node bi-quadratic rectangle, all with one point integration for the pressure.
- nine-node bi-quadratic and composite of four equal four node bilinear elements with three point integration for the pressure.

Hughes and Malkus [15] presented a general equivalence theory between RIP and mixed principles. The potential of this equivalence lies in the essence that the convergence criteria and error bounds developed for mixed principles will be readily available for RIP. The equivalence theory, however, is shown to be applicable for linear plane stress and strain problems and is not generalized to axisymmetric and 3D cases.

2.3.3. Average constraint approach

In this approach the constraint function, Eq. (2.2), is weakened on the element level by choosing a constraint space with less DOF than the actual problem. The cnstraint is then averaged by integrating over the element level and the penalty function is introduced [16]. This amounts to finding a stationary point $\boldsymbol{u}$ for the functional

$$W_a = W + (1/2\varepsilon)\left[\int_V (1/V)\int_V (I_3 - I)\right]^2 \mathrm{d}V. \tag{2.19}$$

In [17] averaging of the constraint is dealt by projecting the function over the basis of the constraint space which has less DOF than those of the actual problem.

The method is directly applicable to lower as well as higher-order elements and seems to be independent of element distortion and of the amount of strain. The finite element treatment of the method is rather lengthy but not complicated for implementation in existing programs [2, 18].

2.3.4. Bubble function approach

Introducing a bubble function to an element shape function will have the effect of increasing the DOF of the primary variables and the freedom of having these to be different from the DOF of the auxiliary variable [2, 3]. This is similar to *weakening* of the constraint equation (2.2). The advantage of such approach is the availability of the procedure in most finite element commercial programs. The main concern is the extension to nonlinear analysis. In [2] two procedures are described for such extension. In both approaches, the nodeless DOF are condensed and recovered in each nonlinear iteration. Also a special iteration loop in the stress calculation routine is described to eliminate any external of residual forces associated with the nodeless DOF.

2.4. Numerical examples

2.4.1. Inflation of a circular plate

A simply supported flat plate subjected to an external follower pressure of 45 psi at its bottom surface is analyzed; shown as in Fig. 1. A Mooney–Rivlin material model is assumed with $C_1 = 80$ psi and $C_2 = 20$ psi. An initial Poisson's ratio of 0.499 is assumed to simulate nearly incompressible behavior. The problem is highly nonlinear and the response departs quickly from the linear analysis. Various formulations considered are: 20 eight-node axisymmetric solid with reduced integration, 40 four-node axisymmetric solid with average constraint, 20 four-node axisymmetric solid with average constraint and 20 four-node axisymmetric solid with bubble function. The load deflection curves based on the above formulations are given in Fig. 2 and compared to the results given by Oden [6] and based on mixed field formulation. Very good agreement is indicated. It is noted that the problem did not converge for pressures beyond 43 psi pressures. As may be seen from the displacement shape, Fig. 3, at this pressure elements became highly distorted resulting in near zero area and poor conditioning of the Jacobian matrix. It is noted that, in this example, at the high end of the pressure range, the bubble function approach tends to consume more iterations than the average constrained one. It is also noted that increasing Poisson's ratio to 0.4999 or 0.49999 substantially increases the number of iterations and in some cases caused divergence of the solution. It is important to report, however, that such increase in

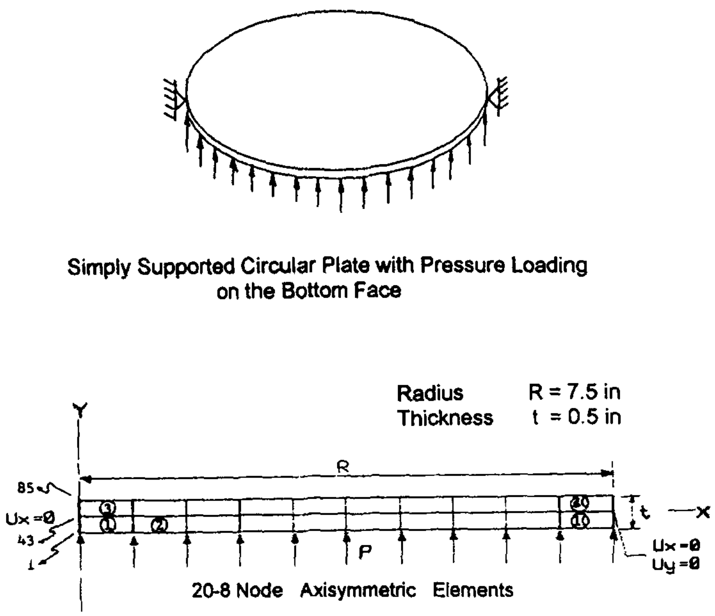

Fig. 1. Geometry and finite element model of the circular plate.

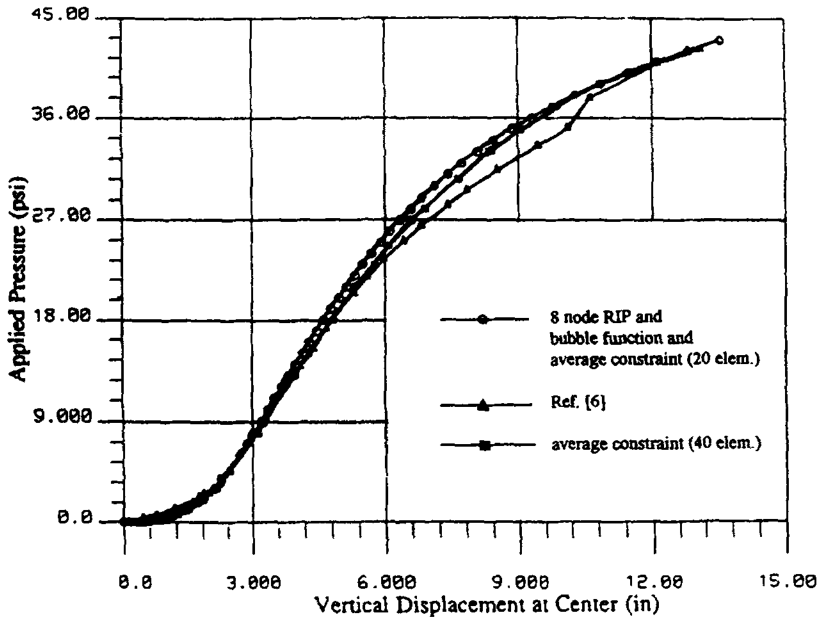

Fig. 2. Pressure-central deflection curve for the circular plate.

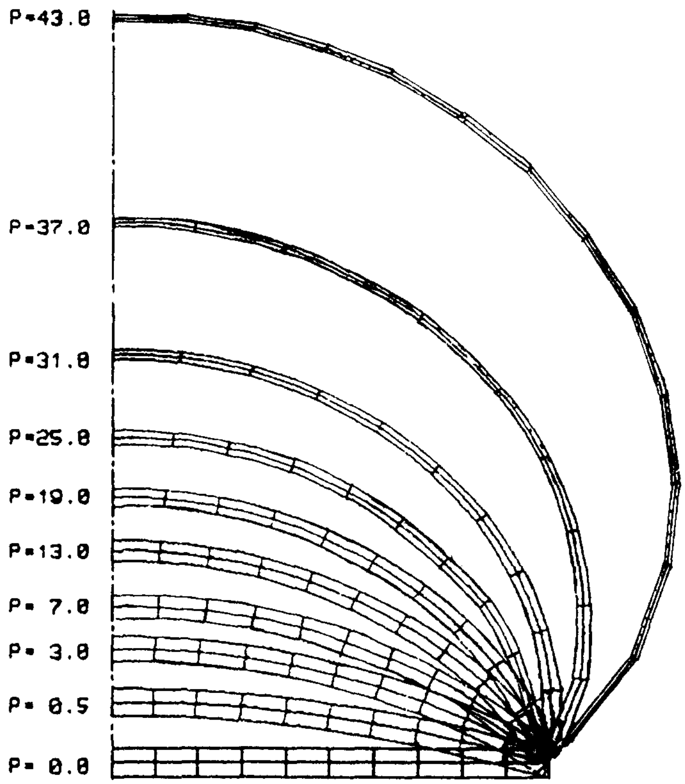

Fig. 3. Deformation shapes at various pressure levels.

Poisson's ratio did not substantially affect the results and, at the most, showed less than 3% difference in the predicted displacements.

3. Elastoplastic materials

3.1. Infinitesimal strain case

In order to study the elastoplastic behavior of a material, one should address three fundamental points; yield criteria, hardening rule and plastic stress-strain relation. In this section, we only discuss the last point.

In the plastic range, additive decomposition of the total strain increments into elastic and plastic increments is assumed such that

$$d\varepsilon_{ij}^{\text{tot}} = d\varepsilon_{ij}^{\text{e}} + d\varepsilon_{ij}^{\text{p}}. \tag{3.1}$$

The elastic constitutive relations establish the elastic strains as

$$d\varepsilon_{ij}^{\text{e}} = C_{ijkl}^{-1} d\sigma_{kl}, \tag{3.2}$$

where C_{ijkl}^{-1} is the inverse of the elastic constitutive tensor. The plastic strain components are assumed to be proportional to the gradient of the plastic potential function, Q, which is the same as the yield function, f, for associated plasticity, such that

$$\mathrm{d}\varepsilon_{ij}^{\mathrm{p}} = \mathrm{d}\lambda \frac{\partial f}{\partial \sigma_{ij}}, \tag{3.3}$$

where $\mathrm{d}\lambda$ is the proportionality constant. To calculate the proportionality factor $\mathrm{d}\lambda$, we start by a general form for the yield function as

$$f = f(\sigma_{ij}, k, T, \dot{\varepsilon}), \tag{3.4}$$

where σ_{ij} are the stress components, k is a hardening parameter, T is the temperature, and $\dot{\varepsilon}$ is the strain rate. It may be shown that for such case, the expression for the proportionality constant $\mathrm{d}\lambda$ may take the following form:

$$\mathrm{d}\lambda = (1/A)\left[\frac{\partial f}{\partial \sigma_{ij}} C_{ijkl}^{\mathrm{e}} \left\{ \mathrm{d}\varepsilon_{ij}^{\mathrm{tot}} - \alpha_{kl}(\mathrm{d}T) - \frac{\partial [C_{klmn}]^{-1}}{\partial T}\sigma_{mn}(\mathrm{d}T) - \frac{\partial [C_{klmn}]^{-1}}{\partial \dot{\varepsilon}}\sigma_{mn}(\mathrm{d}\dot{\varepsilon}) \right\} + \left(\frac{\partial f}{\partial T}\mathrm{d}T + \frac{\partial f}{\partial \dot{\varepsilon}}\mathrm{d}\dot{\varepsilon}\right)\right], \tag{3.5}$$

where

$$A = \left[\frac{\partial f}{\partial \sigma_{ij}} C_{ijkl}^{\mathrm{e}} \frac{\partial f}{\partial \sigma_{kl}} - \frac{\partial f}{\partial k}\frac{\partial f}{\partial \varepsilon_{kl}^{\mathrm{p}}}\frac{\partial f}{\partial \sigma_{ij}}\right]. \tag{3.6}$$

The above relations constitute the basic equations for characterizing the behavior of an elastoplastic material with the assumption that strains are small or infinitesimal. Most of the existing commercial finite element codes employ the above relations as one of their material models in the material library [18, 19]. Various yield functions and hardening rules are available as standard options.

3.2. Large strain case

Intensive effort has been made to describe elastoplastic material behavior at finite strains but little of this effort has become widely accepted, see for example Green and Naghdi [20, 21]. If an infinitesimal theory is used to describe the material behavior at finite strains with the stress replaced by the second Piola–Kirchhoff stress and the infinitesimal strains replaced by Green–Lagrange strain tensor, then a special case of the theory of Green and Naghdi is obtained. In this approach, appropriate tensor transformations may be applied to otain the constitutive relations in other formulations, e.g., updated Lagrangian or Eulerian. The computational complications arising from the accurate utilization of the theory of Green and Naghdi and other similar approaches have so far made the use of these theories unfeasible in finite element programs.

Another widely used approach for large strains is to employ an infinitesimal relation written in an invariant form with respect to rigid body motion [22–24]. In this approach, we may still assume a linear decomposition of strain increments into elastic and plastic parts. Then, the finite element formulation leads to an equation of the form

$$d\sigma_{ij} = C_{ijkl} d\varepsilon_{kl}, \tag{3.7}$$

where the constitutive tensor components are now function of the current state of yielding and work hardening. With the restriction of isotropic hardening and small recoverable strains, these components may be formed in the same way as in the small strain analysis. To generalize the equation to large strain analysis, a suitable frame indifferent stress rate is employed. Two specific examples of such rates are the Truesdell and Jaumann ones. In these two cases, it may be shown that the final constitutive relation takes the following form.

Using Truesdell rate [25],

$$\Delta S_{ij} = \left|\frac{\partial \mathbf{X}}{\partial \mathbf{X}}\right| \frac{\partial X_i}{\partial X_m} \frac{\partial X_j}{\partial X_n} \frac{\partial X_r}{\partial X_k} \left\{ C_{mnkl} \frac{\partial X_s}{\partial X_l} + \sigma_{mn} \frac{\partial X_s}{\partial X_k} - \sigma_{mk} \frac{\partial X_s}{\partial X_n} - \sigma_{nk} \frac{\partial X_s}{\partial X_m} \right\} \Delta E_{rs}. \tag{3.8}$$

Using Jaumann rate [24],

$$\Delta S_{ij} = \left|\frac{\partial \mathbf{X}}{\partial \mathbf{X}}\right| \frac{\partial X_i}{\partial X_m} \frac{\partial X_j}{\partial X_n} \frac{\partial X_r}{\partial X_k} \left\{ C_{mnkl} \frac{\partial X_s}{\partial X_l} + \sigma_{mk} \frac{\partial X_s}{\partial X_n} + \sigma_{nk} \frac{\partial X_s}{\partial X_m} \right\} \Delta E_{rs}. \tag{3.9}$$

3.3. Anisotropic models/composites applications

Most of the existing theories describing anisotropic material behavior emerged from those describing isotropic models. Fundamental problems are inherent to such approach and many basic questions have to be addressed. Among these are the proper extension of yield and plastic potential functions, appropriate flow rule and appropriate integration scheme for the constitutive relation [25–31].

With regard to the yield criteria, Hill's criteria [25] is probably the simplest and, at the same time, has definite physical meaning and provides good agreement with experimental results for certain class of materials [31]. This criteria may be expressed in the following form:

$$f = F(\sigma_{22} - \sigma_{33})^2 + G(\sigma_{33} - \sigma_{11})^2 + H(\sigma_{11} - \sigma_{22})^2 + 2L\sigma_{23}^2 + 2M\sigma_{31}^2 + 2N\sigma_{12}^2 - 1, \tag{3.10}$$

where the axes 1, 2 and 3 coincide with the principal axes of anisotropy and F, G, H, L, M and N are material constants characterizing the current state of anisotropy. Hill's criteria assumes no difference between the yield stress in tension and compression, i.e., no Bauschinger effect. To eliminate this restriction some modifications have been proposed, e.g., Shih and Lee [26] proposed the following criteria:

$$3f = M_{ij}\sigma_i\sigma_j - L_i\sigma_i - K, \tag{3.11}$$

where M_{ij} are material constants describing the variation of the yield function with orientation and L_i are constants giving difference in tension and compression. In Eq. (3.11) σ_1 stands for σ_{11} and σ_2 stands for σ_{22}, etc.

This modification increases the number of anisotropic material constants and complicates the use of these criteria. Phenomenological criteria based on experimental data or physical modeling, see, for example, [30] may provide a better alternative. Although such phenomenological yield functions do not result directly from microstructure-based calculations, they have some advantages over yield functions calculated from the crystallographic texture of polycrystalline aggregates. Namely, they are easy to implement, specially in finite element environment, easy to adapt to different materials and lead to faster computations.

Hecker reviewed numerous critical experiments [32] to assess the yield surface shape, found that normality rule was never violated. Barlat [30], assuming anisotropic material possessing three mutually orthogonal planes of symmetry, proposed a yield function, which is convex and independent of hydrostatic pressure. The yield function is

$$f=(3I_2)^{m/2}\left\{\left[2\cos\left(\frac{2\theta+\pi}{6}\right)\right]^m+\left[2\cos\left(\frac{2\theta-3\pi}{6}\right)\right]^m+\left[-2\cos\left(\frac{2\theta+5\pi}{6}\right)\right]^m\right\}-2\bar{\sigma}^m, \tag{3.12}$$

where

$$\theta=\arccos(I_3/I_2^{3/2}),$$

$$I_2=\frac{(fF)^2+(gG)^2+(hH)^2}{3}+\frac{(aA-cC)^2+(cC-bB)^2+(bB-aA)^2}{54},$$

$$I_3=\frac{(cC-bB)(aA-cC)(bB-aA)}{54}+fghFGH$$

$$-\frac{(cC-bB)(fF)^2+(aA-cC)(gG)^2+(bB-aA)(hH)^2}{6}$$

$$\begin{bmatrix}\frac{cC-bB}{3} & hH & gG\\ hH & \frac{aA-cC}{3} & fF\\ gG & fF & \frac{bB-aA}{3}\end{bmatrix}=\begin{bmatrix}S_{xx} & S_{xy} & S_{xz}\\ S_{yx} & S_{yy} & S_{yz}\\ S_{zx} & S_{zy} & S_{zz}\end{bmatrix}=S_{ij}.$$

Here, S_{ij} is stress deviator, and a, d, c, f, g and h are six anisotropic coefficients. Three of them can be obtained from three stresses in the directions of the symmetry axes and the other three coefficients are derived from the shear yield stress. The exponent m can take any real value larger than one (1) but practically, m should be larger than six (6), depending on the severity of the texture, the resulting yield function reduces to the isotropic case when the constant coefficients a, b, c, f, g and h are all equal to 1, and particularly to Tresca for $m = 1$ and von Mises for $m = 2$.

It is generally accepted, however, that yield criteria for anisotropic and composite materials are not yet well established. Furthermore, most of the above developments assume transverse anisotropy with three orthogonal planes of symmetry that is not applicable to many composites. The development in the area of anisotropic flow rule and efficient method for integrating the constitutive relations follows closely the trend in the development of yield criteria. Basic concepts like an effective stress–strain relations for highly anisotropic and composite materials is thought to be unacceptable [29]. The form of the effective stress–strain curve is determined experimentally using a specific load path and then assumed to be valid for general multi-axial loading. While this assumption is generally acceptable for isotropic materials, it is shown that the tangent modulus of the effective stress–strain is generally load path-dependent for highly anisotropic and composite materials.

Some of the main concerns in the finite element side of the problem are the complexity of the criteria and the computer time required to perform a practical type of analysis specially for composites with large number of layers. Large number of constants in the criteria will also impose serious limitations on the practical use and calculation of these constants. Most of the existing commercial finite element programs use Hill's criteria which, as discussed above, has many limitations in highly anisotropic and composite materials.

As mentioned above, one of the important aspects to be considered in anisotropic yield criteria is the number of material constants to be specified in two- and three-dimensional analyses. Some developments impose specific dependency of these constants through pre-specified relation to account for plastic incompressibility [19]. It is felt that certain averaging procedures of the incompressibility condition may eliminate unphysical imposed relations on the material constants [18].

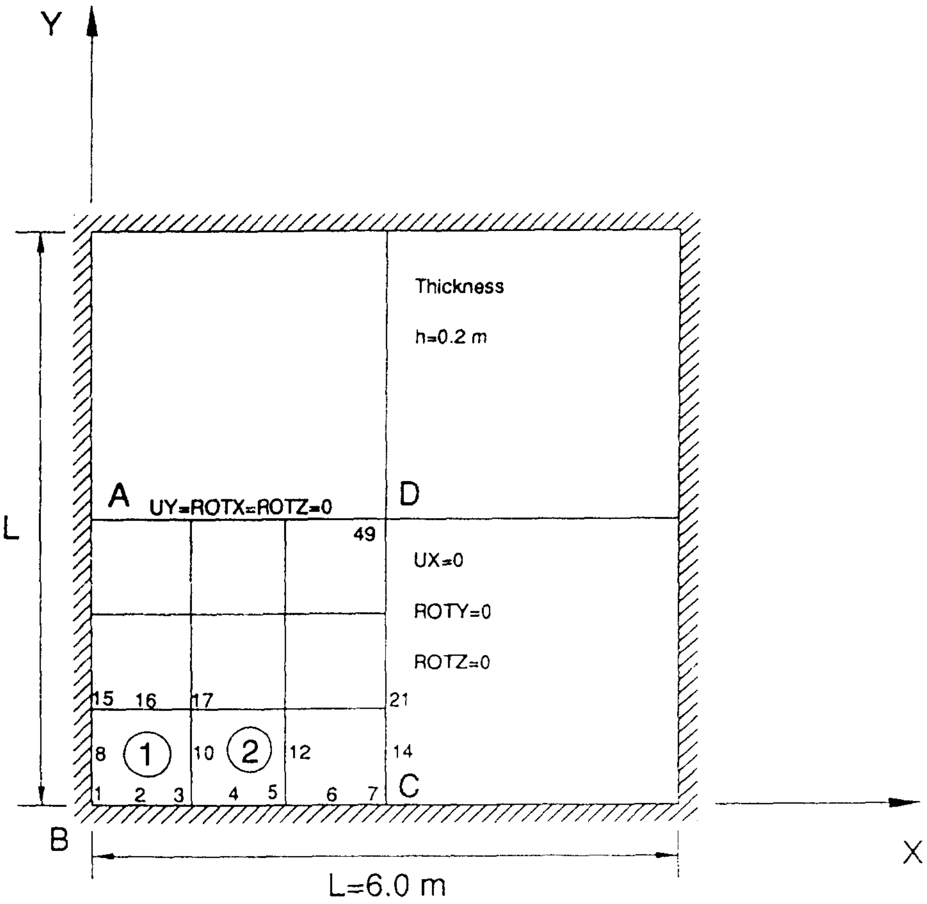

Fig. 4. Clamped plate model.

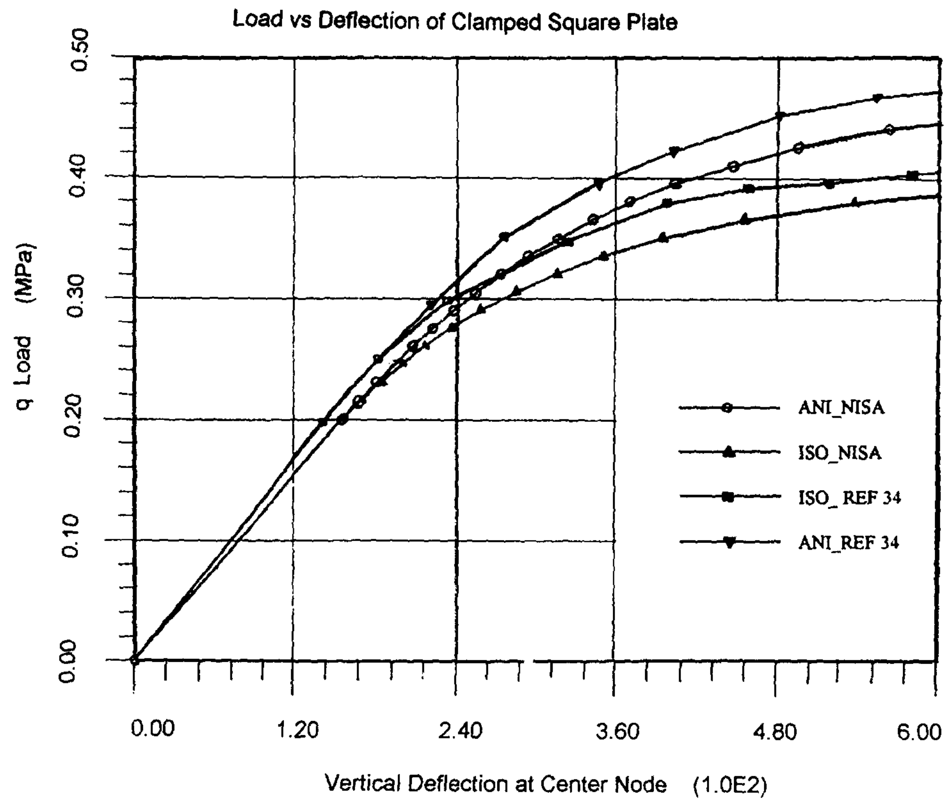

Fig. 5. Load deflection curve of clamped square plate.

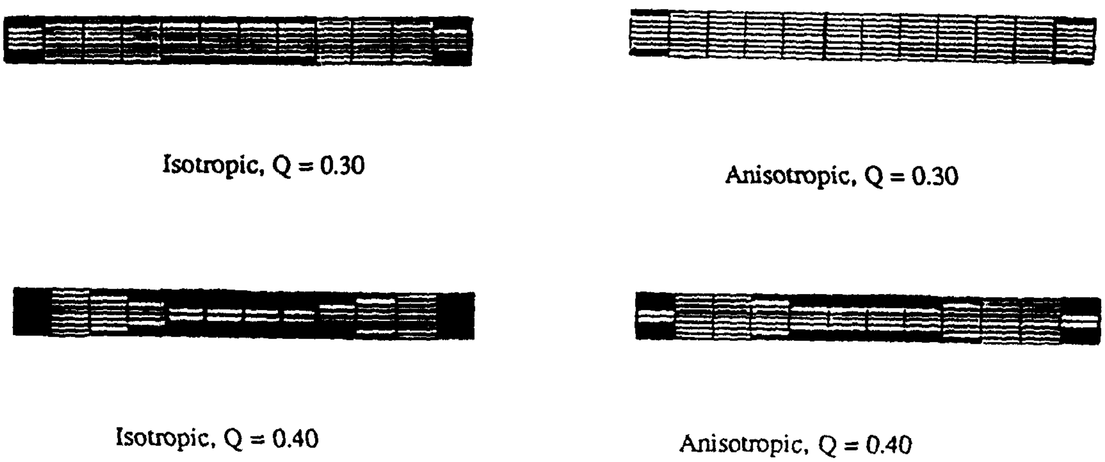

Fig. 6. Spread of plastic zone through the thickness of the plate.

3.4. Numerical example

A clamped square plate subjected to uniform pressure q is analyzed [18]. The geometry and dimensions are shown in Fig. 4. Both isotropic and anisotropic cases are considered. Hill's yield function is used. Anisotropy is introduced by assuming different yield strengths in the principal material directions.

The vertical displacement at the center (point D) of the plate is plotted in Fig. 5 for various load levels and compared to the results in [33], for both isotropic and anisotropic cases. The spread of plastic zone at various load levels are plotted in Fig. 6.

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