Approximate solutions of non-singular linear differential equation using Bernstein operational matrix of differentiation

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Abstract

In this paper, exact and approximate analytical solutions of a non-singular linear differential equations are obtained by the Bernstein operational matrix of differentiation. Different from other numerical techniques, Bernstein polynomials and their properties are employed for deriving a general procedure for forming this matrix. In The present paper we used Bernstein operational matrix to solve a linear non-singular boundary and initial value problems it was solved by wavelet analysis method. We report our numerical finding and compare it with Wavelet method. Our results become more accurate.

1 Introduction

In this work, we consider the non-singular problems of the type

$$y''(x) + p(x)y'(x) + q(x)y(x) = g(x) \quad 0 < x \leq 1, \tag{1}$$

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subject to the boundary conditions

$$y(0) = \alpha_1, \quad y(1) = \beta_1, \quad \text{or} \quad y'(0) = \alpha_2, \quad y(0) = \beta_2, \tag{2}$$

where p, q and g are continuous functions on $(0, 1]$ and the parameters $\alpha_1, \alpha_2, \beta$ are real constants.

Problems of form (1)–(2) have been studied in many areas of science, chemistry and physics for example equilibrium of isothermal gas sphere, reaction-diffusion process, geophysics, etc. Exact/approximate solutions of these problems are of great importance due to its wide application in scientific research. Singular BVPs have been studied by several authors. Nasab and Kilicman [4] used Wavelet analysis method for solving linear and nonlinear singular boundary value problems. Bataineh et al. [1] used Legendre Operational matrix to approximate solution of two points.

Bernstein operational matrix of differentiation, proposed by Bhatti and Bracken [2] used Bernstein polynomial Basis to solve Differential Equation. Pandey and Kumar [5], is Bernstein operational matrix for solving Lane-Emden type equations. Yousefi and Behroozifar [7], used Operational matrices of Bernstein polynomials and their applications to solve Bessel differential equation. Recently, Yuzbasi [8], used Bernstein polynomials to solve fractional riccati type differential equations. Rasit and Sezer [6], employed Bernstein series to solve class of lane-Emden equation.

In The present paper we used Bernstein operational matrix to solve a linear and nonlinear singular boundary value problems it was solved by wavelet analysis method Nasab and Kilicman [4]. We report our numerical finding and compare it with Wavelet method. Our results become more accurate, we can see only small number of Bernstein polynomial basis functions are needed to get the approximate solution with which is full agreement with the exact solution up to 10 digits. This article is structured as follows. In Section 2, we describe the basic formulation of Bernstein polynomials and its operational matrix differentiation. In section 3, we explain the applications of the operational matrix of derivative. In section 4, we report our numerical finding and compare it with Wavelet method , exact solution and demonstrate the validity, accuracy and applicability of the operational matrices by considering numerical examples. Section 5, consist of brief summary and conclusion.

2 Bernstein polynomials and its operational matrix of differentiation

The Bernstein polynomials of degree m are defined by

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \quad i = 0, 1, \ldots, m$$

where the binomial coefficient is

$$\binom{m}{i} = \frac{m!}{i!\,(m-i)!}.$$

There are $m+1$ nth-degree Bernstein polynomials. For mathematical convenience, we usually set $B_{i,m} = 0$, if $i < 0$ or $i > m$.

In general, we approximate any function $y(x)$ with the first $(m+1)$ Bernstein polynomials as

$$y(x) = \sum_{i=0}^{m} c_i B_{i,m}(x) = \mathrm{C}^{\mathrm{T}} \phi(x), \tag{3}$$

where

$$\mathrm{C}^{\mathrm{T}} = [\mathrm{c}_0, \mathrm{c}_1, \ldots, \mathrm{c}_m],$$

$$\phi(x) = [B_{0,m}(x), B_{1,m}(x), \ldots, B_{m,m}(x)]^{\mathrm{T}}.$$

The derivatives of the vector $\phi(x)$ can be expressed as

$$\frac{\mathrm{d}\phi(x)}{\mathrm{d}x} = \mathrm{D}^1 \phi(x) \tag{4}$$

where D^1 is the $(m+1) \times (m+1)$ operational matrix of derivative given as.

Which is satisfy Eq.(4), also it can be generalized Eq.(4) as

$$\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} = (\mathrm{D}^1)^2 \phi(x), \ldots, \frac{\mathrm{d}^n\phi(x)}{\mathrm{d}x^n} = (\mathrm{D}^1)^n \phi(x). \tag{5}$$

For example with $m = 4$

$$D = \begin{pmatrix} -4 & -1 & 0 & 0 & 0 \\ 4 & -2 & -2 & 0 & 0 \\ 0 & 3 & 0 & -3 & 0 \\ 0 & 0 & 2 & 2 & -4 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix},$$

and if $m = 5$

$$D = \begin{pmatrix} -5 & -1 & 0 & 0 & 0 & 0 \\ 5 & -3 & -2 & 0 & 0 & 0 \\ 0 & 4 & -1 & -3 & 0 & 0 \\ 0 & 0 & 3 & 1 & -4 & 0 \\ 0 & 0 & 0 & 2 & 3 & -5 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{pmatrix}.$$

3 Applications of the operational matrix of derivative

To solve (1)–(2) by means of the operational matrix of derivative Yousefi and Behroozifar [7], we approximating $(y(x))^n$ and $g(x)$ by Bernstein Polynomials as

$$\begin{aligned} (y(x))^n &\simeq \left(\mathrm{C}^{\mathrm{T}}\phi(x)\right)^n, & (6)\\ g(x) &\simeq \mathrm{G}^{\mathrm{T}}\phi(x), & (7) \end{aligned}$$

where the vector $\mathrm{G}^{\mathrm{T}} = [g_0(x), \ldots, g_m(x)]^{\mathrm{T}}$ represent the non-homogenous term. By using Equations(4), (6) and (7) we have

$$\begin{aligned} y''(x) &\simeq \mathrm{C}^{\mathrm{T}}(\mathrm{D}^1)^2\phi(x), & (8)\\ y'(x) &\simeq \mathrm{C}^{\mathrm{T}}\mathrm{D}^1\phi(x), & (9) \end{aligned}$$

Employing equations (6)–(9) the residual $\Re(x)$ for Eq.(1) can be written as

$$\Re(x) \simeq \mathrm{C}^{\mathrm{T}}(\mathrm{D}^1)^2\phi(x) + p(x)\mathrm{C}^{\mathrm{T}}\mathrm{D}^1\phi(x) + q(x)\left(\mathrm{C}^{\mathrm{T}}\phi(x)\right)^n - \mathrm{G}^{\mathrm{T}}\phi(x). \qquad (10)$$

Now, to find the solution $y(x)$ given in Eq. (3) we have two cases

3.1 Linear case

For $n = 1$, we generate $m - 1$ linear equations as in a typical tau method Canuto et al. [3] by applying

$$\int_0^1 \Re(x)P_j(x)\,\mathrm{d}x = 0, \quad j = 0, 1, \ldots, m-2. \qquad (11)$$

Also, by substituting boundary conditions (2) into Equations (6) and (9) we have

$$u(0) = \mathrm{C}^{\mathrm{T}}\phi(0) = \alpha_1, \quad u(1) = \mathrm{C}^{\mathrm{T}}\phi(1) = \beta \qquad (12)$$

or

$$u'(0) \quad = \quad \mathrm{C}^{\mathrm{T}}\mathrm{D}^{1}\phi(0) = \alpha_2, \quad u(1) = \mathrm{C}^{\mathrm{T}}\phi(1) = \beta \tag{13}$$

Equations (11)–(13) generate (m+1) set of linear equations respectively. These linear equations can be solved for unknown coefficients of the vector C. Consequently, $u(x)$ given in Eq. (6) can be easily calculated.

3.2 Nonlinear case

For $n = 2, 3, \ldots$ we find all intersection points between $p_i(x)$ and $p_{i-2}(x), i = 2, 3, \ldots, m-2$ in Bernstein polynomials then we have $(m-2)$ points call collocation points. We substitute collocation points in Equations (10)-(12), we have $m-2$ of equations the unknowns are c_i. Nonlinear equations which can be solved using Newton's iterative method. Consequently $y(x)$ given in Eq. (3) can be calculated. This method (collocation method) to avoid the difficulty of integration.

4 Numerical experiments

To illustrate the effectiveness of the presented method, we shall consider the following examples of two-point BVPs.

Example 1.

We first consider the linear two-point BVP, which has been considered by Nasab and Kilicman [4],

$$y'(x) + 4y(x) = f(x), \tag{14}$$

$$y(0) = 0, \quad y(1) = 1, \tag{15}$$

where $f(x)$ is defined by

$$f(x) = \begin{cases} 4x^2 + 2x, & 0 \le x < \frac{1}{2}, \\ 4x + 1, & , \quad \frac{1}{2} \le x \le 1. \end{cases}$$

The exact solution of this problem is

$$y(x) = \begin{cases} x^2, & 0 \le x < \frac{1}{2}, \\ x & , \quad \frac{1}{2} \le x \le 1. \end{cases}$$

By applying the technique of Section 3.1, with $m = 3, 5$ and 8, we have

$$c^T\phi(x) = c_0 p_{0,3}(x) + c_1 p_{1,3}(x) + c_2 p_{2,3}(x) + c_3 p_{3,3}(x). \tag{16}$$

According to Eq. (5), we have

$$\mathrm{D}^1 = \begin{pmatrix} -3 & -1 & 0 & 0 \\ 3 & -1 & -2 & 0 \\ 0 & 2 & 1 & -3 \\ 0 & 0 & 1 & 3 \end{pmatrix}, \tag{17}$$

and using Eq. (11) we have

$$\frac{1}{14}c_0 + \frac{41}{70}c_1 + \frac{37}{140}c_2 + \frac{11}{140}c_3 - \frac{1}{6} = 0, \tag{18}$$

$$\frac{-1}{70}c_0 + \frac{12}{35}c_1 + \frac{57}{140}c_2 + \frac{37}{140}c_3 - \frac{2}{5} = 0 \tag{19}$$

Now by applying Eq. (12), we have

$$c_0 = 0 \tag{20}$$

$$c_3 = 1. \tag{21}$$

Finally by solving Equations (18)–(21) we get

$$c_0 = 0, \quad c_1 = 0, \quad c_2 = \frac{1}{3}, \quad c_3 = 1. \tag{22}$$

Thus

$$y(x) = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 - 3x + 3x^2 - x^3 \\ 3x - 6x^2 + 3x^3 \\ 3x^2 - 3x^3 \\ x^3 \end{pmatrix} = x^2$$

Which is the exact solution.

Now for the second part of example, that $f(x) = 4x + 1$ by applying same steps we have

$$\frac{1}{14}c_0 + \frac{41}{70}c_1 + \frac{37}{140}c_2 + \frac{11}{140}c_3 - \frac{9}{20} = 0, \tag{23}$$

$$\frac{1}{70}c_0 + \frac{12}{35}c_1 + \frac{57}{140}c_2 + \frac{37}{140}c_3 - \frac{13}{20} = 0. \tag{24}$$

Now applying Eq. (12), we have

$$c_0 = 0 \tag{25}$$

$$c_3 = 1, \tag{26}$$

Finally by solving Equations (23)–(26) we get

$$c_0 = 0, \quad c_1 = \frac{1}{3}, \quad c_2 = \frac{2}{3}, \quad c_3 = 1. \tag{27}$$

Thus

$$u(x) = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 1 \end{pmatrix} \begin{pmatrix} 1 - 3x + 3x^2 - x^3 \\ 3x - 6x^2 + 3x^3 \\ 3x^2 - 3x^3 \\ x^3 \end{pmatrix} = x,$$

which is the exact solution.

Example 2.

Consider the Bessel differential equation of order zero,

$$xy''(x) + y'(x) + xy(x) = 0 \tag{28}$$

$$y(0) = 1, \quad y'(0) = 0. \tag{29}$$

A solution known as the Bessel function of the first kind of order zero denoted by J_0 is

$$J_0(x) = \sum_{q=0}^{\infty} \frac{(-1)^q}{(q!)^2} \left(\frac{x}{2}\right)^{2q}. \tag{30}$$

We solve this problem using Bernstein polynomials with $m = 8$ as can be seen in Table 1, only small number of Bernstein polynomials are needed to get the approximate solution which is in full agreement with the exact solution up to 10 Digits. We compare our result with Nasab and Kilicman [4] paper which got the answer is in full agreement with the exact solution just up to 9 Digits.

Example 3.

Consider the singular boundary value problem

$$y''(x) + |4x - 1|y'(x) - 32 = 8(4x - 1)|4x - 1|, \tag{31}$$

$$y(0) = 1, \quad y(1) = 9 \tag{32}$$

which has the exact solution $y(x) = (4x - 1)^2$. By applying the technique described in (11) with $m = 2$ we obtain

$$c_0 = 1, \qquad c_1 = -3, \qquad c_2 = 9. \tag{33}$$

Thus we can writ

$$y(x) = (4x - 1)^2, \tag{34}$$

5 Conclusions

In this paper, the Bernstein operational matrix of derivative was applied to solve a class of non-singular point . Different from other numerical techniques, only small size operational matrix is required to provide the solution at high accuracy. This matrix is used to approximate numerical solution of a class of non-singular point. It can be clearly seen in the paper that the proposed method is working well even on few numbers of terms of the Bernstein polynomials.

References

[1] A.S. Bataineh, A.K. Alomari, and I. Hashim. Approximate solutions of singular two-point BVPs using Legendre operational matrix of differentiation. *Journal of Applied Mathematics*, 2013:Art. ID 547502, 2013.

[2] M.I. Bhatti and P. Bracken. Solutions of differential equations in a Bernstein polynomial basis. *Computational and Applied Mathematics*, 205:272–280, 2007.

[3] C. Canuto, M.Y. Hussaini, A. Quarteroni, and T.A. Zang. *Spectral Method in Fluid Dynamics*. Prentice Hall, New Jersey, 1988.

[4] A.K. Nasab and A. Kilicman. Wavelet analysis method for solving linear and nonlinear singular boundary value problems. *Applied Mathematical Modelling*, 37:5876–5886, 2013.

[5] K. Pandey and N. Kumar. Solution of Lane-Emden type equations using Bernstein operational matrix of differentiation. *New Astronomy*, 17:303–308, 2012.

[6] O. Rasit and M. Sezer. Bernstein series solution of a class of Lane-Emden type equations. *Mathematical Problems in Engineering*, vol. 2013:Art. ID 423797, 2013.

[7] S.A. Yousefi and M. Behroozifar. Operational matrices of Bernstein polynomials and their applications. *International Journal of System Science*, 41:709–716, 2010.

[8] S. Yuzbasi. Numerical solution of fractional Riccati type differential equations by means of the Bernstein polynomials. *Applied Mathematics and Computation*, 219:6328–6343, 2013.

Table 1: The values of the unknown x for $m = 8$ of Example 2.

x_i	Present work	Wavelet analysis [4]	Exact solution
0.1	0.9975015620	0.997501562	0.9975015620
0.2	0.9900249722	0.990024972	0.9900249722
0.3	0.9776262465	0.977626246	0.9776262465
0.4	0.9603982266	0.960398226	0.9603982266
0.5	0.9384698072	0.938469807	0.9384698072
0.6	0.9120048634	0.912004863	0.9120048634
0.7	0.8812008886	0.881200888	0.8812008886
0.8	0.8462873527	0.846287352	0.8462873527
0.9	0.8075237981	0.807523798	0.8075237981
1.0	0.7651976865	0.765197686	0.7651976865